Evaluating allocations of opportunities*

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Abstract

This paper provides a robust criterion for comparing lists of probability distributions - interpreted as allocations of opportunities - faced by different social groups. Borrowing from decision making under objective ambiguity, we argue in favour of comparing those collections of probability distributions on the basis of a uniform - among groups - valuation of the expected utility associated to these distributions. We identify an empirically implementable criterion for comparing these lists of probability distributions - conic extension of Zonotope inclusion - that is agreed upon by all conceivable such valuations that exhibit aversion toward inequality of opportunities. We illustrate our criterion by evaluating allocations of educational opportunities among castes and genders in different Indian states.

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1 Introduction

Improving and equalizing opportunities are considered by many to be important social objectives. In the US, opinion surveys conducted by the Pew research center[1] in the last 25 years have consistently found that 90% of respondents agree that "our society should do what is necessary to make sure that everyone has an equal opportunity to succeed". This "equal opportunity to succeed" ideal is commonly interpreted as meaning that individuals' probabilities (chances) of reaching outcomes of interest should not depend on *morally irrelevant* characteristics such as skin color, gender, national origin, family background and so on. This view therefore refers to a partition of the society into groups, formed on some exogenous morally irrelevant basis, who face different probabilities of achieving outcomes of interest. This paper proposes a general and robust methodology for comparing these lists of probability distributions - *allocations of opportunities* as we shall call them - in a way that is sensitive to both the average probability of reaching "favorable" outcomes offered to the members of the group and the equalization of these probabilities among groups.

To illustrate and motivate our approach, consider Figure 1 below that shows the probabilities that low and high caste adults aged between 30 and 40 in two neighboring states of India (West-Bengal and Odisha) achieve one of the six educational levels reported in the 68th round of the Employment-Unemployment survey of the Indian National Sample Survey Organization (NSSO). These Education levels are ranked from illiteracy (1) to upper tertiary education (6). Low caste status is defined as belonging to the official category of Scheduled Caste (SC) and Scheduled Tribe (ST), the remainder being considered high caste. Figure 1 clearly shows that low caste adults have significantly lower chance of achieving any education level than high caste adults whichever state they live in.

Yet one may want to go beyond the mere observation that educational opportunities are unequally distributed among caste groups in both states. One may, in

[1] see e.g. https://www.pewresearch.org/2011/03/11/the-elusive-90-solution.

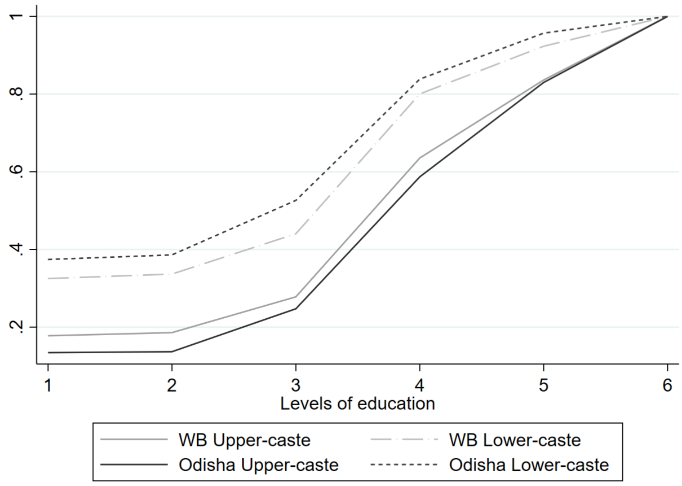


Figure 1: Cumulative distribution of education levels, low and high caste groups, Odisha and West Bengal, 2012.

particular, want to make comparative statements on the extent by which this caste inequality in educational opportunities differs between West Bengal and Odisha. Figure 1 suggests indeed that educational opportunities faced by low and high caste adults - clearly unequal in both West Bengal and Odisha - are "more unequal" in the latter than in the former. In effect, the risk of low caste adults failing to reach any education level is higher in Odisha than in West Bengal. However, an opposite ranking is observed for high caste adults who are more at risk of failing to reach any level of education in West Bengal than in Odisha. Hence, the unprivileged group does better in West Bengal than in Odisha while the opposite is true for the favored group. Since the average distribution of educational opportunities, calculated symmetrically between high and low caste adults is quite - albeit not perfectly - similar, it could be concluded that caste-based inequalities of educational opportunities are larger in Odisha than in West Bengal. Of course pure equalization may not be the only aim of opportunities-sensitive policies. It may also be important, just as in the case of income distributions, to increase the opportunities offered to some, or to all, individuals *in addition to* equalizing them.

Comparing alternative collections of probability distributions over outcomes is conceptually similar to comparing decisions under objective ambiguity examined for example in Ahn (2008), Gravel, Marchant, and Sen (2012) or Gravel, Marchant, and Sen (2012). Exploiting this similarity, our approach proposes to compare collections of probability distributions by means of the uniform expected valuation of expected utility criterion. This criterion works by assigning to every outcome a utility level, and by evaluating a probability distribution by its expected utility. Alternative lists of probability distributions are then compared on the basis of an expected valuation of these expected utilities, under the assumption that all social groups are equally likely (or carry equal ethical worth). If the evaluation of allocations of opportunities exhibits aversion to opportunities inequality, the valuation must be a concave function of the expected utility. The main contribution of this paper is an empirically operational test that identifies when one allocation of opportunities among groups is better than another for all comparisons of allocations of opportunities made by inequality averse uniform expected valuations of expected utility criteria. The test, explained in detail in the paper, is the inclusion of the *quasi-ordering extended zonotopes* uniquely associated with the allocations under comparison. The zonotope of any list of probability distributions is the set of all Minkowski sums of those probability distributions (see e.g. Koshevoy (1995), Koshevoy (1998)). A quasi-ordering extended zonotope is a zonotope that has been enlarged with translations that capture the assumptions made about the ranking of outcomes of interest to groups' members. Our approach is, indeed, quite general in that respect and is applicable to any possible set of outcomes: ordered, non-ordered or in-between these two extremes. While the quasi-ordering extended zonotope inclusion test is theoretically implementable with any number of groups, its actual implementation may sometimes be difficult. However, we are able to provide a precise and finite test for the general criterion in many cases, one of them being the two-group case discussed above. Another is the case, very commonly considered in the inequality of opportunity literature, where distributions over outcomes are ordered across groups as per expected utility. In this case, our criterion works in a sequential fashion just like the well-known generalized Lorenz criterion (Shorrocks (1983)). In the complete ordering of outcomes case, the test consists indeed in checking for first order dominance between the worst off

- in expected utility - groups, then for the (uniform average) of the two worst-off groups, and so on.

We also illustrate our criterion by comparing allocations of educational opportunities offered to Indian adults depending upon their caste and gender in a few Indian states. While our criterion is, just like Lorenz dominance, *a priori* incomplete, it nonetheless succeeds in conclusively comparing a majority of the Indian states in terms of caste and gender. While the obtained ranking of the states often mirrors their ranking in terms of their average overall opportunities, there are a few states where inequality of educational opportunities is so great that even their relatively favorable average distribution of opportunities does not enable them to dominate other states with a lower but equally distributed average. An interesting example of this is Kerala, an Indian state that is often favorably portrayed on the education front. Yet, unequal distribution of educational opportunities among castes prevents Kerala from dominating - as per our criterion - the state of Andhra Pradesh whose average distribution of educational opportunities is dominated at the first-order of stochastic dominance.

1.1 Related literature

There is an abundant literature on equality of opportunities that has been nicely surveyed in Roemer and Trannoy (2016), and, in its theoretical underpinnings, by Fleurbaey (2008). This literature stands on what John Roemer (1996) calls the Dworkin (1981) "cut" between the characteristics that affect an individual's outcome for which the individual should be held *responsible* and the morally irrelevant characteristics that determine what this literature calls the individual's "type". The main creed of this literature is that opportunity equalization should be concerned with *equalizing outcomes* among individuals who share the same "responsibility characteristics". However, no attempt should be made to equalize outcomes that can be shown to result from the free exercise of responsibility alone. In recent years, the cut between the variables affecting individuals' achievements has been enlarged to luck and randomness (see e.g. Vallentyne (2002), Lefranc, Pistolesi, and Trannoy (2009) and Lefranc and Trannoy (2017)). Along with a few other contributions (e.g. Bénabou and Ok (2001) and Mariotti and Veneziani (2017)), our approach departs from this cut inspired literature by being agnostic about individuals' degree of responsibility for some of their characteristics. Responsibility actually plays no role in our approach, even though individuals in each group may be considered "responsible" for their success in the life (defined in our approach by their probability of achieving the outcomes of relevance). Another important difference between our approach and those surveyed by Roemer and Trannoy (2016) is that we provide a robust dominance-based definition of opportunity equalization and improvement, while many others seek to define - somewhat binarily - either inequality or (perfect) equality of opportunity. The family of uniform concave valuations of expected utility criteria on which we base our dominance has however been proposed as a possible social objective in some of the theoretical literature on equality of opportunity, notably the one inspired by the so-called ex ante approach to inequality of opportunity examined by Van De Gaer (1993), Ooghe, Schokkaert, et al. (2007), and, in the context of intergenerational mobility measurement, by Martinez, Schockkaert, and VandeGaer (2001). However, these authors have not provided dominance criteria over the family of their social objectives. Most contributions to the literature that define opportunity equalization in terms of the Dworkin (1981) cut (like Peragine (2004) and other contributions surveyed by Brunori, Ferreira, and Peragine (2021)), tend to proceed by decomposing total outcome inequality (measured by some index) into *within* and *between* group inequality (see e.g. Ferreira and Gignoux (2011)), defining between-group inequality, in the tradition of Shorrocks (1984), with respect to the group' mean outcome. A focus on group mean outcome is also a feature of the approach developed by Bénabou and Ok (2001) in the specific context of mobility measurement. Yet, focusing on group mean outcome is restrictive because it disregards all information related to the possibly varying riskiness of those outcomes across groups.

The contributions to the literature that we find the closest to our are, for different reasons, Andreoli, Havne, and Lefranc (2019) and Mariotti and Veneziani (2017). Andreoli, Havne, and Lefranc (2019) propose a robust definition of opportunity *equalization* across types based on a sequential (if more than two types are involved) comparison of the absolute value of the "gaps" between probability distributions faced by individuals of two types. Their criterion is robust because it applies to allocations of opportunities where the probability distributions faced by differing types are ordered by a large class of preferences. However, by requiring such a unanimous agreement over the ranking of the different types' probability distributions (for example by first-order stochastic dominance), Andreoli, Havne, and Lefranc (2019) restrict the applicability of their criterion to those allocations of opportunities where this unanimous agreement holds. We do not impose this restriction on the criterion examined herein. Moreover, while the criterion proposed by Andreoli, Havne, and Lefranc (2019) is sensitive to welfare *gaps* between distributions, it is not sensitive to welfare *levels*. As a result, their criterion may promote a policy that actually reduces the opportunities offered to every group if the reduction in opportunities is not uniform across groups and reduces the gap between them. By contrast, our criterion is sensitive to both the "levels" of opportunities offered to the groups and the gap between them, and incorporates in its very definition a trade-off between aggregate improvement in opportunities and unequal sharing of those improvements among groups. This trade-off is similar in spirit to that underlying the generalized Lorenz curve (see e.g. Shorrocks (1983)) in conventional one-dimensional income inequality measurement.

Mariotti and Veneziani (2017) propose a justification for comparing allocations of opportunities with only two ordered outcomes (say good and bad) on the basis of the product - over all groups - of the probabilities of occurrence of the good outcome. When applied to such a restricted setting, their specific complete ranking of allocations of opportunities is compatible with our incomplete one. However our criterion, which happens to be the intersection of a very large class of ranking of which the Mariotti and Veneziani (2017) ranking is only one member, applies to all allocations of opportunities and is therefore not restricted to those whose outcomes are binary.

1.2 Organization of the paper

The remainder of this paper proceeds as follows. The next section describes the general setting in which allocations of opportunities are evaluated and provides a normative foundation for this evaluation from the stand point of an ethical observer placed behind the veil of ignorance. Section 3 describes the operational extended zonotope criterion and establishes its equivalence with the ranking of allocations of opportunities made by all opportunity-inequality-averse uniform expected valuation of expected utility. Section 3 also indicates how the criterion can be empirically implemented in a large number of cases. Section 4 presents the results of the empirical implementation of the criterion for appraising gender- and caste-based inequalities of educational opportunities in India, and Section 5 concludes.

2 A framework for evaluating allocations of opportunities

We are interested in comparing allocations of opportunities to members of some exogenously given groups. These groups may be based on caste, religion, race, gender, family background, etc. They may also be based on their members having exerted the same level of *responsibility* (if this idea is subscribed to). We do not assume the number of such groups to be the same across allocations. For instance, we may consider allocation of opportunities among one group alone. Our approach would then view each of such allocations as achieving (trivially) perfect equality of opportunities, even though they may differ in the "average level" of those opportunities. At the other extreme, we may consider cases where every individual forms a distinct group. If this latter interpretation is favored, we could then interpret a collection of individuals with identical observable characteristics (gender, background, etc.) - what is usually considered as a "group" - as the replication of as many identical individuals as there are members in this collection.

The opportunities offered to a group are described from an *ex ante* point of view[2] by the *probabilities* of its members achieving *relevant outcomes*. We assume specifically that there are k such outcomes taken from the set $\{1, ..., k\}$. Outcomes are to be interpreted as anything observable that people have reason to value, like income, education or health levels (expressed in discrete units). Outcomes may also be formed of pairs of, say, education and health levels. Hence, our approach does not require outcomes to be completely ordered. We can even take the extreme view that they are not ordered at all. For example, if we were to address the allocation, among males and females, of the opportunity to join the army, there would be only two outcomes (joining or not) in no obvious order. Formally, the ranking of outcomes may be viewed as resulting from some quasi-ordering $\geq_{QO}$ on $\{1, ..., k\}$ with the interpretation that $j \geq_{QO} h$ if and only if outcome j is "clearly better" for an agent than outcome h. The (extreme) case of incomplete quasi-ordering where *none* of the outcomes can be compared with one another is denoted by $\geq_{\emptyset}$. The other extreme case of *complete* ordering commonly assumed in the equality of opportunity literature surveyed in Roemer and Trannoy (2016) is denoted by $\geq_C$. Given the quasi-ordering $\geq_{QO}$, we assume that the outcomes $\{1, ..., k\}$ are labeled in a way compatible with $\geq_{QO}$: if $h < i$ then $h \geq_{QO} i$ *does not* hold. We also denote by $\mathcal{U}^{\geq_{QO}} \subset \mathbb{R}^k$ the set of all lists of utility numbers compatible with the quasi-ordering $\geq_{QO}$ defined by:

$$\mathcal{U}^{\geq_{QO}} = \{(u_1, ..., u_k) \in \mathbb{R}^k : \; j \geq_{QO} h \Longrightarrow u_j \geq u_h, \;\; \forall j, h \in \{1, ..., k\}\} \tag{1}$$

Any allocation of opportunities $\mathbf{p}$ is depicted as an $n(\mathbf{p}) \times k$ row-stochastic matrix:

$$\mathbf{p} = \begin{bmatrix} p_{11} & \cdots & p_{1k} \\ \cdots & \cdots & \cdots \\ p_{n(\mathbf{p})1} & \cdots & p_{n(\mathbf{p})k} \end{bmatrix}$$

[2]See Fleurbaey (2010) and Fleurbaey (2018) for discussions of, and alternative approaches to, the normative analysis of "socially risky situations".

where p_{ij}, for $i = 1, ..., n(\mathbf{p})$ and $j = 1, ..., k$, denotes the probability that an agent from group i will achieve outcome j in allocation $\mathbf{p}$ and $n(\mathbf{p})$ denotes the number of groups in $\mathbf{p}$. For any allocation $\mathbf{p}$, we denote by p_i the probability distribution (opportunities) faced by group i in $\mathbf{p}$ and by $\overline{p}$ its (symmetric across groups) average defined by:

$$\overline{p} = \frac{1}{n(\mathbf{p})} \sum_{i=1}^{n(\mathbf{p})} p_i.$$

We denote by $\mathbb{A} = \bigcup_{n \geq 1} \left(\Delta^{k-1}\right)^n$ the set of all conceivable allocations.

Allocations of opportunities are to be compared by an *ethical observer* who agrees with the ranking of the outcomes provided by $\geq_{QO}$, and who is placed behind a veil of ignorance as to the group to which she would belong if she lived in the societies described by these allocations. Such an ethical observer would compare allocations by means of some ordering $\succsim$, with asymmetric and symmetric factors $\succ$ and $\sim$. Since the ordering $\succsim$ is defined on the whole set $\mathbb{A}$, it is in particular defined on the set Δ^{k-1} of all conceivable one-group allocations and, therefore, on all probability distributions over the k outcomes. Hence the ethical observer is also a decision maker under risk who is capable of ordering all probability distributions over outcomes. We specifically focus on ethical observers who use an ordering $\succsim$ for which there is a (continuous) increasing function $\Phi : \mathbb{R} \to \mathbb{R}$ and k real numbers $u_1, ..., u_k \in \mathcal{U}^{\geq_{QO}}$ such that:

$$\mathbf{q} \succsim \mathbf{p} \Longleftrightarrow \frac{1}{n(\mathbf{q})} \sum_{i=1}^{n(\mathbf{q})} \Phi\left(\sum_{j=1}^{k} q_{ij} u_j\right) \geq \frac{1}{n(\mathbf{p})} \sum_{i=1}^{n(\mathbf{p})} \Phi\left(\sum_{j=1}^{k} p_{ij} u_j\right) \tag{2}$$

for any two allocations of opportunities $\mathbf{p}$ and $\mathbf{q}$ in $\mathbb{A}$. An ordering satisfying this property *could* therefore be thought of as resulting from the following three-step procedure:

1. In the first step, every group in the compared allocations is given an *expected utility* that results from the assignment of utility numbers $u_1, ..., u_k$ to outcomes in a way that reflects their ranking by the quasi-ordering.

2. In the second step, this expected utility is assigned a *valuation* by the ethical

observer through some function Φ.

3. In the third step, the ethical observer ranks the allocations on the basis of the *expected valuation* of their groups' expected utilities, under the assumption that the ethical observer is equally likely to fall into any group.

We accordingly call *Uniform expected valuator of expected utility* (UEVEU) any such ethical observer. There are actually quite a few of them, as many in fact as there are logically conceivable valuation functions Φ and logically conceivable ways of assigning utility to outcomes in a manner consistent with $\geq_{QO}$.[3] While we believe that evaluating allocations of opportunities by means of an ordering that is numerically representable as per Expression (2) is plausible, it is possible thanks to results in Gravel, Marchant, and Sen (2011, 2012) to single out this family of rankings as the only one that satisfy a collection of natural axioms.[4]

We take the view that UEVEU ethical observers exhibit aversion to inequality of opportunity, which we define as a preference for an allocation exhibiting *no* inequality of opportunity - say because it consists of one single group - over allocations exhibiting some inequality of opportunity. In order to define this notion precisely, we first introduce the following definition of comparative aversion to opportunity inequality among ethical observers.

Definition 1 *We say that* $\succsim_1$ *exhibits* at least as much aversion to inequality of opportunity *as* $\succsim_2$ *if, for every probability distribution* $\rho \in \Delta^{k-1}$ *and allocation* $\mathbf{p} \in \mathbb{A}$*, we have* $\rho \succsim_2 \mathbf{p} \Longrightarrow \rho \succsim_1 \mathbf{p}$.

In words, ethical observer 1 exhibits at least as much aversion to inequality of opportunity as ethical observer 2 if any preference of the latter for a perfectly equal allocation of opportunities as compared to an arbitrary allocation of opportunities is also endorsed by the former. With this notion of comparative aversion

[3]Evaluating opportunities by means of a UEVEU criterion has been suggested by Martinez, Schockkaert, and VandeGaer (2001) (see their Equations (1)-(3) p. 528) in connection with mobility measurement. It can also be observed that comparing alternative lists of probability distributions on the basis of Inequality (2) is what the criterion characterized by Ahn (2008) in the context of decision making under objective ambiguity would recommend, under the additional assumption that all social groups are equally likely to form.

[4]The axioms and the characterization result are provided in the online Appendix A.

to inequality of opportunity, we can define aversion to inequality of opportunity *tout court* as the fact of being more averse to inequality of opportunity than an ethical observer who is neutral with respect to it. This requires however a definition of neutrality with respect of opportunity inequality. In the usual income inequality setting, neutrality to income equality is defined as the fact of considering as equivalent all income distributions with the same per capita income. While we are not aware of the existence of a well-accepted standard of neutrality toward inequality of opportunity, we believe that a plausible candidate would be considering as equivalent all allocations that distribute among their groups the same (symmetric) average probability distribution over outcomes. Hence we could say that $\succsim$ exhibits *neutrality to equality of opportunity* if $\mathbf{p} \sim \mathbf{q}$ for any two allocations $\mathbf{p}$ and $\mathbf{q}$ in $\mathbb{A}$ such that $\overline{p} = \overline{q}$. With this standard of neutrality, we can define aversion to inequality of opportunity as follows.

Definition 2 *An ordering* $\succsim$ *on* $\mathbb{A}$ *is said to exhibit* aversion to inequality of opportunity *if there exists an ordering* $\succsim_0$*, exhibiting neutrality to inequality of opportunity, such that* $\succsim$ *exhibits at least as much aversion to inequality of opportunity as* $\succsim_0$.

As shown in Gravel, Marchant, and Sen (2012) (Proposition 5), the notion of comparative aversion to opportunity inequality can translate, when expressed for orderings represented by (2), into a statement of "comparative concavity" applied to the function Φ of that expression.[5] Combining this fact with Definition 2, we can therefore establish the following.

Proposition 1 *An ordering* $\succsim$ *on* $\mathbb{A}$ *that can be numerically represented as per (2) exhibits aversion to inequality of opportunity if and only if* Φ *is concave.*

The normative dominance approach that we use for comparing allocations is

[5]In the case of two groups and two ordered outcomes, an intuitive interpretation of the role played by concavity is that an opportunity inequality averse ethical observer should prefer the allocation of opportunities where each group has a $1/2$ probability of achieving either outcomes to the allocation where one group achieves for sure the best outcome while the other group achieves for sure the worst outcome.

based on a consensus among all opportunity inequality averse UEVEU ethical observers. This dominance as defined as follows.

Definition 3 *Given a quasi-ordering* $\geq_{QO}$ *on* $\{1, ..., k\}$*, we say that allocation* $\mathbf{q}$ *UEVEU-dominates allocation* $\mathbf{p}$*, which we denote by* $\mathbf{q} \succsim_{UEVEU}^{QO} \mathbf{p}$*, iff*

$$\frac{1}{n(\mathbf{q})}\sum_{i=1}^{n(\mathbf{q})}\Phi\left(\sum_{h=1}^{k}q_{ih}u_h\right) \geq \frac{1}{n(\mathbf{p})}\sum_{i=1}^{n(\mathbf{p})}\Phi\left(\sum_{h=1}^{k}p_{ih}u_h\right) \tag{3}$$

for all increasing and concave functions $\Phi : \mathbb{R} \to \mathbb{R}$ *and all* $(u_1, ..., u_k) \in \mathcal{U}^{\geq_{QO}}$*.*

We find worth emphasizing that, except in the case where no assumption is made on the ranking of outcomes, this dominance criterion combines a concern for reducing opportunity inequality with an objective of improving the opportunities offered to some, or to all, of the groups. This is particularly clear when outcomes are completely ordered, in which case any allocation of opportunities - regardless how unequal it is - will be considered weakly better than a perfectly egalitarian allocation in which the members of all groups are sure to obtain the worst outcome (a situation referred to as *hell* by Mariotti and Veneziani (2017)). In the other direction, any allocation of opportunities will be considered by a UEVEU criterion to be worse than the egalitarian allocation that gives to anyone the certainty to end in the best outcomes.

3 An operational criterion for evaluating allocations of opportunities

3.1 The criterion in the general case

From now on, we focus on allocations of opportunities among the same number of groups so that $n(\mathbf{p}) = n(\mathbf{q}) = n$ for some integer $n \geq 2$. The value of this n is clearly immaterial for an inequality such as (3) and we therefore abstract from it. Even with a fixed number of groups, the number of functions Φ and combinations of utility numbers for which Inequality (3) needs to be verified in order to establish whether or not UEVEU dominance occurs between two distributions is too large to make the criterion usable in practice. In this section, we identify an equivalent formulation of UEVEU criterion that significantly eases this verification. We start by observing that $\mathcal{U}^{\geq_{QO}}$ is a non-empty closed convex cone, the dual[6] of which is defined by

$$\mathcal{U}_*^{\geq_{QO}} := \{(v_1, ..., v_k) \in \mathbb{R}^k : \sum_{j=1}^{k} v_j u_j \geq 0 \text{ for all } (u_1, ..., u_k) \in \mathcal{U}^{\geq_{QO}}\}. \quad (4)$$

Note that $\mathcal{U}_*^{\geq_{QO}} = \{\mathbf{0}^k\}$ if and only if none of the outcomes can be compared with one another. Another remark about the dual cone is that all the $k-$tuples $(v_1, ..., v_k)$ that it contains have components that sum to 0. We state this formally as follows.[7]

Remark 1 *Let* $(v_1, ..., v_k) \in \mathcal{U}_*^{\geq_{QO}}$ *for some quasi-ordering* $\geq_{QO}$ *of* $\{1, ..., k\}$. *Then* $v_1 + ... + v_k = 0$.

The dual cone associated with $\mathcal{U}^{\geq_{QO}}$ is the set of all changes in the probability distribution over outcomes that increase expected utility for all utility functions compatible with $\geq_{QO}$. In plain English, it is the set of all clear improvements in opportunities to achieve the outcomes. This interpretation is supported by the fact that the sum of these changes is zero so that they produce a new probability distribution over outcomes which cumulates to 1, just like the initial distribution. The exact nature of these changes in the distribution depends on the precise definition of the quasi-ordering.

The proposed operational definition of opportunity equalization makes use of the zonotope set $\mathbf{Z}(\mathbf{p}) \subset \mathbb{R}_+^k$ associated with any allocation of opportunities $\mathbf{p} \in \mathbb{A}$,

[6] Which is the negative of what Rockafellar (1970) p. 121 calls the *polar* of $\mathcal{U}^{\geq_{QO}}$.

[7] See the online appendix for the proofs of all auxiliary results, such as lemmas, remarks and propositions.

and defined by:

$$\mathbf{Z}(\mathbf{p}) := \left\{ z \in \mathbb{R}_+^k : z = \sum_{i=1}^{n(\mathbf{p})} \theta_i p_i \text{ for some } \theta_i \in [0,1],\ i = 1, \ldots, n \right\} \quad (5)$$

A closely related set has been used by Koshevoy (1995) (see also Koshevoy and Mosler (1996)) to define a criterion called by this author *Lorenz majorization.* We use the zonotope defined by (5) to define what we call Quasi-Ordering Extended Zonotope (QOEZ) dominance as follows.

Definition 4 *We say that allocation* $\mathbf{q}$ *QOEZ dominates allocation* $\mathbf{p}$*, which we write as* $\mathbf{q} \succsim_Z^{QO} \mathbf{p}$*, if* $Z(\mathbf{q}) + \mathcal{U}_*^{\geq QO} \subseteq Z(\mathbf{p}) + \mathcal{U}_*^{\geq QO}$.

In plain English, $\mathbf{q}$ QOEZ dominates $\mathbf{p}$ if any Minkowski sum of distributions of probabilities observed in the groups in $\mathbf{q}$, possibly modified by a transformation in $\mathcal{U}_*^{\geq QO}$ that unambiguously improves expected utility, is also a Minkowski sum of distributions of probabilities in $\mathbf{p}$, again possibly modified by an expected utility improving transformation in $\mathcal{U}_*^{\geq QO}$. While Minkowski sums of distributions may superficially resemble weighted averages of those distributions - and therefore be evocative of an “equalizing operation” - this is not really so because the weights θ_i included in their definition (as per (5)) do not sum to 1 and may, in particular, only consist of 0 and 1. There is however an alternative definition of the zonotope, stated in the following lemma, that makes the *equalization operation* underlying extended Zonotope dominance more apparent.

Lemma 1 $\mathbf{Z}(\mathbf{p}) = Co \left\{ \sum_{i=1}^{n} \alpha_i p_i : (\alpha_1, ..., \alpha_n) \in \{0,1\}^n \right\}$*, for every* $\mathbf{p} \in \mathbb{A}$.

Hence the zonotope of an allocation of opportunities $\mathbf{p}$ is the *convex hull* of all the possible *partial sums* of the probability distributions over outcomes offered to the members of the groups. This zonotope contains therefore all weighted averages of those partial sums. Viewed in this way, the QOEZ dominance of allocation $\mathbf{p}$ by allocation $\mathbf{q}$ roughly means that all probability distributions in $\mathbf{q}$, possibly modified by expected utility improvements in $\mathcal{U}_*^{\geq QO}$ or all their partial sums, or all averages of the same are simply improvements over weighted average of probability distributions in $\mathbf{p}$. Koshevoy (1998) and Koshevoy and Mosler (2007) have also introduced, in a different context, a similar-looking criterion based on the inclusion of extended zonotope sets. However, the extensions that they consider, based on the multiplications of k-dimensional commodity bundles by their prices, are quite different from ours.

We now establish the main equivalence between the ranking of two allocations of opportunities as per QOEZ dominance and the ranking of those allocations agreed upon by all opportunity-inequality-averse UEVEU ethical observers who compare outcomes by means of the quasi-ordering $\geq_{QO}$.

Theorem 1 *The two following statements are equivalent:*

(*i*) $\mathbf{q} \succsim_Z^{QO} \mathbf{p}$;

(*ii*) $\mathbf{q} \succsim_{UEVEU}^{QO} \mathbf{p}$.

Theorem 1 thus provides a rather solid justification for comparing allocations of opportunities on the basis of QOEZ dominance. A simple, but interesting, implication of the QOEZ dominance of one allocation by another is the corresponding domination, by all conceivable expected utilities, of their respective average distributions. In effect:

Remark 2 *If* $\mathbf{q} \succsim_Z^{QO} \mathbf{p}$ *then* $\overline{q} - \overline{p} \in \mathcal{U}_*^{\geq_{QO}}$.

Let us now interpret Theorem 1 in the two extreme cases where no outcomes are comparable, and where all outcomes are ordered as per their rank in the set $\{1, ..., k\}$. Starting with the first case, and combining standard results on one-dimensional inequality measurement and Theorem 3.1 in Koshevoy and Mosler (1996), we easily obtain the following remark.

Remark 3 *The two following statements are equivalent:*

(*i*) $Z(\mathbf{q}) \subset Z(\mathbf{p})$ *and* $\overline{p} = \overline{q}$;

(ii) $\mathbf{q} \succsim^{\emptyset}_{UEVEU} \mathbf{p}$.

Turning now to the case where outcomes are completely ordered from the worst (1) to the best (k), the lists of utility numbers $(u_1, ..., u_k) \in \mathbb{R}^k$ for which unanimity is sought are those that satisfy $u_1 \leq u_2 \leq ... \leq u_k$. It cannot be excluded that such a list of utility numbers is of uncountably infinite size. A similar critic applies to the result in Theorem 1, where the set $\mathcal{U}^{\geq_{QO}}$ of lists $(u_1, ..., u_k)$ of utility numbers compatible with $\geq_{QO}$ with respect to which the dual cone $\mathcal{U}_*^{\geq_{QO}}$ of changes $(v_1, ..., v_k)$ in the distribution - which must be added to the Zonotope sets before checking for inclusion - is defined. How can the dual cone of an uncountably infinite set be identified in practice ? In the following lemma, we alleviate this difficulty by showing that for any uncountably infinite set $\mathcal{U}^{\geq_{QO}}$ of lists $(u_1, ..., u_k)$ of utility numbers compatible with $\geq_{QO}$, there is a finite set of lists of utility numbers (each actually taken from the pair $\{0, 1\}$) that generates exactly the same dual cone $\mathcal{U}_*^{\geq_{QO}}$.

Lemma 2 *We have*

$$\mathcal{U}_*^{\geq_{QO}} = \left\{ \mathbf{v} \in \mathbb{R}^k : \sum_{j=1}^{k} v_j u_j \geq 0 \ \forall (u_1, ..., u_k) \in \mathcal{U}^{\geq_{QO}} \cap \{0, 1\}^k \right\}.$$

In particular, this implies that the dual cone of the set $\mathcal{U}^{\geq_C}$ corresponds to the set of changes $(v_1, ..., v_k)$ in the probability distributions that produce *first-order stochastic* improvements over the probability distributions to which they are applied. The dual cone $\mathcal{U}_*^{\geq_C}$ is hence given by:

$$\mathcal{U}_*^{\geq_C} = \left\{ v \in \mathbb{R}^k : \sum_{j=1}^{k} v_j = 0, \sum_{g=h}^{k} v_g \geq 0 \text{ for } h = 2, ..., k \right\}.$$

3.2 Elementary operations

An alternative understanding of QOEZ dominance can be obtained from the underlying *elementary transformations* in the allocations of opportunities that this criterion considers worth doing. While we do not identify all these elementary transformations in the general n-group case - see, however, the results of the next subsection - we identify some of them. We start with the following, also discussed by Kolm (1977) in the more general setting of multidimensional inequality measurement.

Definition 5 (Uniform averaging) *We say that* $\mathbf{q}$ *is obtained from* $\mathbf{p}$ *through a* uniform averaging *operation if there exists an* $n \times n$ *bistochastic matrix* $\mathbf{b}$ *such that* $\mathbf{q} = \mathbf{b}.\mathbf{p}$

This operation consists in *uniformly averaging* the various distributions of outcomes of the different groups. Specifically, if $\mathbf{q}$ is obtained from $\mathbf{p}$ through a *uniform averaging* operation, then for every group i, the probability q_{ih} of someone from that group achieving outcome h is a weighted average of the probabilities of people from the different groups in $\mathbf{p}$ achieving that outcome. This averaging is *uniform* in the sense that, for any group i, the weights used in the calculation of the average do not depend on the outcome.

The second elementary operation that we examine is what we call a *bilateral equalizing transfer*. Contrary to uniform averaging - which does not use information on the ranking of outcomes - the operation of bilateral equalizing transfer relies heavily on such information. The formal definition of this operation is as follows.

Definition 6 (Equalizing transfer) *We say that* $\mathbf{q}$ *is obtained from* $\mathbf{p}$ *through a bilateral equalizing transfer if there exist indices* $i_1,\ i_2,\ i_1', i_2' \in \{1, ..., n\}$ *and* $v \in \mathcal{U}_*^{\geq QO}$ *such that* $p_j = q_j$ *for all* $j \notin \{i_1, i_2, i_1', i_2'\}$*, and*

$$q_{i_1'} = p_{i_1} + v,\ q_{i_2'} = p_{i_2} - v,\ p_{i_2} - p_{i_1} - v \in \mathcal{U}_*^{\geq QO}.$$

A bilateral equalizing transfer is an operation that improves (through some change v) the opportunities faced by a group and that worsens (through the same v applied in reverse) the opportunities faced by another group, in the case where the latter group's opportunities are unambiguously better than the former's regarding the quasi-ordering $\geq_{QO}$ or equivalently, thanks to Donaldson and Weymark (1998), regarding all complete rankings of outcomes whose intersection is $\geq_{QO}$. We observe that such a transformation only concerns two distributions of outcomes in each of the two allocations (and leaves the other groups' distributions unchanged). Hence, by comparison with the uniform averaging operation which concerns the *entire matrix*, a bilateral equalizing transfer, as its name suggests, is a *local operation* that concerns only two rows of each of the matrices under comparison.

Example 1 *Let us illustrate this transformation in the case of an incomplete ranking of outcomes. Consider for this purpose the following binary health-education example where the outcomes are* $(0,0)$ *(bad health, low education),* $(0,1)$ *(bad health, high education)* $(1,0)$ *(good health, low education) and* $(1,1)$ *(good health, high education) with the quasi-ordering* $\geq_{QO}$ *defined by* $(1,1) \geq_{QO} (0,1) \geq_{QO} (0,0)$ *and* $(1,1) \geq_{QO} (1,0) \geq_{QO} (0,0)$ *((*$(0,1)$ *and* $(1,0)$ *being incomparable). Assume that there are only two groups, and consider the two allocations:*

$\mathbf{p} =$

	$0,0$	$0,1$	$1,0$	$1,1$
gr. 1	$1/2$	$1/6$	$1/6$	$1/6$
gr. 2	$1/4$	$1/4$	$1/4$	$1/4$

$\mathbf{q} =$

	$0,0$	$0,1$	$1,0$	$1,1$
gr. 1	$7/16$	$19/96$	$1/6$	$19/96$
gr. 2	$5/16$	$7/32$	$1/4$	$7/32$

Observe first that in allocation $\mathbf{p}$*, the probabilities of achieving the four outcomes in group 1 provides a lower expected utility than those of group 2. This can be seen by the fact that, for any of the two complete rankings of the four outcomes that are consistent with* $\geq_{QO}$*, the distribution of outcomes faced by group 2 first-order stochastically dominates that faced by group 1. A second observation that can be made is that the move from* $\mathbf{p}$ *to* $\mathbf{q}$ *has been done by improving the probability distribution of group 1 by the vector* $v = (-1/16, 1/32, 0, 1/32)$ *and by worsening the probability distribution of group 2 by the corresponding vector* $-v = (1/16, -1/32, 0, -1/32)$*. Note that these two balanced offsetting changes in the distributions of outcomes have preserved the dominance of group 2 over group 1. Yet the spread of that difference has shrunk.*

The last elementary operation that we discuss is not related to reducing inequalities of opportunity. It is rather concerned with improving those opportunities for some, or all, of the groups (up to a permutation of them thanks to the anonymity principle). It is defined as follows.

Definition 7 (Anonymous and Unanimous Expected Utility Improvement) *We say that* $\mathbf{q}$ *is obtained from* $\mathbf{p}$ *through an* anonymous and unanimous expected utility improvement *if there exists a one-to-one function* $\pi : \{1, ..., n\} \longrightarrow \{1, ..., n\}$ *such that, for every* $i \in \{1, ..., n\}$*,* $q_{\pi(i)} = p_{\pi(i)} + v_i$ *holds for some* $v_i \in \mathcal{U}_*^{\geq QO}$.

In words, $\mathbf{q}$ is obtained from $\mathbf{p}$ through an anonymous and unanimous expected utility improvement if there can be a permutation of the groups such that every permuted group in $\mathbf{q}$ has opportunities that provide a greater expected utility than those in $\mathbf{p}$ for all lists of utility numbers compatible with the underlying quasi-ordering. Any such anonymous and unanimous expected utility improvement will clearly be appraised favorably by any UEVEU ethical observer (irrespective of her attitude to opportunity inequality). In the following lemma, we establish formally that performing an equalizing transfer *or* a uniform averaging will also be considered worthwhile elementary operations by opportunity-inequality-averse UEVEU observers.

Lemma 3 *If* $\mathbf{q}$ *is obtained from* $\mathbf{p}$ *through either* uniform averaging *or* equalizing transfer *then* $\mathbf{q} \succsim_{UEVEU}^{QO} \mathbf{p}$.

3.3 Evaluating allocations of opportunities in practice

This section identifies empirical procedures for implementing QOEZ dominance, as well as some difficulties that may arise when using the criterion in general contexts. We start by introducing some additional notation. For any possible number of groups $h = 1, \ldots, n$, let $m(h)$ denote the number of collections of h groups, and label these collections as $J_h^1, ..., J_h^{m(h)}$. With this notation, the following lemma establishes an equivalent, but operationally much simpler, definition of QOEZ dominance.

Lemma 4 $\mathbf{q} \succsim_Z^{QO} \mathbf{p}$ *if and only if, for all* $h = 1, \ldots, n$ *and* $m = 1, \ldots, m(h)$, $\exists \tilde{p} \in Co\left\{\sum_{i\in J_h^1} p_i, \ldots, \sum_{i \in J_h^{m(h)}} p_i\right\}$ *such that* $\sum_{i\in J_h^m} q_i - \tilde{p} \in \mathcal{U}_*^{\geq QO}$.

This lemma says that testing for QOEZ dominance of $\mathbf{q}$ over $\mathbf{p}$ amounts to testing that, for any collection of h groups, one can find a convex combination of the symmetric average of the probability distributions in all collections of h groups in $\mathbf{p}$ that is dominated - as per the quasi-ordering - by the symmetric average probability distribution in the considered collection in $\mathbf{q}$.

It is not difficult to see that the procedure identified by Lemma 4 could lead to an easy verification of the standard non-extended zonotope inclusion test identified in Proposition 3 in the case where there no ranking whatsoever of any two outcomes. However, this test, which can only be applied to allocations of opportunities with the same symmetric average, is unlikely to be of significant empirical interest.

A class of situations of much greater empirical interest that give rise to easy verifications of QOEZ dominance are those where all distributions faced by the different groups can be ordered by all their expected utility consistent with the quasi-ordering of the outcomes. For example, in the Indian illustration discussed in this paper where the complete ordering of education levels is assumed, the various gender- and caste-based groups can all be compared by first-order stochastic dominance. This case has also been considered in some of the empirical literature on measurement of equality of opportunity (see e.g. Andreoli, Havne, and Lefranc (2019) and Lefranc, Pistolesi, and Trannoy (2009)). If the distributions faced by the different groups can all be compared by any expected utility consistent with the quasi-ordering of outcomes, then testing for extended Zonotope inclusion is extremely simple, as shown in the following proposition.

Proposition 2 *For any allocations of opportunities* $\mathbf{q}$ *and* $\mathbf{p} \in \mathbb{A}$ *such that* $q_{i+1} - q_i \in \mathcal{U}_*^{\geq QO}$ *and* $p_{i+1} - p_i \in \mathcal{U}_*^{\geq QO}$ *for all* $i = 1, ..., n-1$, *then* $\mathbf{q} \succsim_Z^{QO} \mathbf{p}$ *if and only if, for any* $h = 1, \ldots, n$

$$\sum_{i=1}^{h} q_i - \sum_{i=1}^{h} p_i \in \mathcal{U}_*^{\geq QO} \tag{6}$$

The test underlying Proposition 2 is reminiscent of the sequential logic underlying generalized Lorenz dominance. Indeed, the test works as follows. First, it compares the worst-off group's distribution in the two allocations. If these distributions are not comparable, then the test fails and the two allocations can not be compared. If one distribution dominates the other, then the average cumulative distribution of the two worst-off groups are compared, and so on. When applied to the complete ordering of outcomes, the procedure described in Proposition 2 bears some resemblance to the test proposed by Dardanoni (1993) (see his Theorem 1) in the context of mobility measurement. Dardanoni (1993) has proposed his test in the context of ranking (squared) mobility matrices with fixed margins in which children from low-background parents face worse opportunities - as appraised by first-order dominance - than children from high-background parents. The test in Proposition 2 can be seen as an extension of his test to the case where the joint distributions represented by the matrices do not have the same margins and where, possibly, $n \neq k$.

An interesting particular case, considered in Mariotti and Veneziani (2017), to which Proposition 2 applies is when $k = 2$ (bad and good outcomes). In such a case, all distributions can be compared by first order dominance. Our criterion would then evaluate allocations of opportunities by comparing sequentially the symmetric average probability of ending in the bad outcome in the j groups for which this probability is the largest in their respective allocation, for any $j = 1, \dots, n$. An allocation of opportunities in which this symmetric average probability of ending in the bad outcome calculated over the j groups where it is the highest is lower than in another allocation for all j groups, would be said to dominate this latter allocation. Non-comparability of the two allocations would obtain if the symmetric average probability of ending in the bad outcome calculated over the j groups where it is the highest is lower in one allocation than in another for some j, but when an opposite conclusion holds for some other number - l say - of groups with the highest probability of ending in the bad outcome. The incomplete ordering of two allocations generated by this criterion is a subrelation of the complete criterion of Mariotti and Veneziani (2017).

There are, however, cases where an "easy" procedure for verifying QOEZ dominance is not readily available. The following example exhibits one of them.

Example 2 *Consider the following two allocations:*

$\mathbf{q} =$

	a	b	c	d
gr. 1	0.025	0.375	0.35	0.25
gr. 2	0.15	0.2	0.35	0.3
gr. 3	0.1	0.35	0.2	0.35

$\mathbf{p} =$

	a	b	c	d
gr. 1	0	0.3	0.6	0.1
gr. 2	0	0.6	0	0.4
gr. 3	0.3	0	0.3	0.4

$$q_1 - \left(\frac{1}{2}p_1 + \frac{1}{3}p_2 + \frac{1}{6}p_3\right) \in \mathcal{U}_*^{\geq QO};$$
$$q_2 - \left(\frac{1}{3}p_1 + \frac{1}{6}p_2 + \frac{1}{2}p_3\right) = 0;$$
$$q_3 - \left(\frac{1}{6}p_1 + \frac{1}{2}p_2 + \frac{1}{3}p_3\right) = 0;$$
$$q_1 + q_2 - \left(\frac{1}{3}(p_1 + p_2) + \frac{1}{2}(p_1 + p_3) + \frac{1}{6}(p_2 + p_3)\right) \in \mathcal{U}_*^{\geq QO};$$
$$q_1 + q_3 - \left(\frac{1}{2}(p_1 + p_2) + \frac{1}{6}(p_1 + p_3) + \frac{1}{3}(p_2 + p_3)\right) \in \mathcal{U}_*^{\geq QO};$$
$$q_2 + q_3 - \left(\frac{1}{6}(p_1 + p_2) + \frac{1}{3}(p_1 + p_3) + \frac{1}{2}(p_2 + p_3)\right) = 0;$$
$$q_1 + q_2 + q_3 - (p_1 + p_2 + p_3) \in \mathcal{U}_*^{\geq QO}.$$

Hence, by Lemma 4, $\mathbf{q} \succsim_Z^C \mathbf{p}$. *However, it would be difficult to identify the candidates for stochastic domination by the various combinations of sums of the* q_i*'s. Take for instance the first dominance. The set* $\{\tilde{p} \in Co(p_1, p_2, p_3) : \ q_1 \succsim^{1st} \tilde{p}\}$ *is equal to* $\{\lambda p_1 + \mu p_2 + (1 - \lambda - \mu)p_3 : \ \lambda \geq 1/2, \ \mu \geq 1/3, \ \lambda + \mu \leq 11/12\}$*, and thus is a convex subset of the interior of* $Co(p_1, p_2, p_3)$*. It is not clear how an element from this set can be found by a procedure based on the verification of a finite number of inequalities.*

3.4 Two-group allocations

A case of significant practical and theoretical interest is when there are only two groups. In such a case, we can implement QOEZ dominance by the finite, and somewhat simple, procedure of majorization, by each of the two distributions of the dominating allocation, of some weighted average of the two distributions of the dominated allocation exactly in the spirit of Lemma 4. While Example 2 shows how Lemma 4 can sometimes be difficult to apply if there are more than two groups, the difficulty vanishes if there are only two groups. To see how the procedure of Lemma 4 works in this case, consider the family $\mathcal{F}^{\geq_{QO}}$ of sets whose elements form a chain with respect to the quasi-ordering $\geq$. This family is formally defined by:

$$\mathcal{F}^{\geq_{QO}} = \{J \subset \{1, ..., k\} : h \in J \text{ and } j \geq_{QO} h \Longrightarrow j \in J\}$$

This family is closely related to the dual cone $\mathcal{U}_*^{\geq_{QO}}$ of the quasi-ordering $\geq_{QO}$ which can indeed be defined, thanks to Lemma 2, by:

$$\mathcal{U}_*^{\geq_{QO}} = \left\{ v \in \mathbb{R}^k : \sum_{h=1}^{k} v_h = 0 \text{ and } \sum_{h \in J} v_h \geq 0 \text{ for all } J \in \mathcal{F}^{\geq_{QO}} \right\} \tag{7}$$

The family $\mathcal{F}^{\geq_{QO}}$ is important because it provides the complete (and finite) list of sets of outcomes whose increases in likelihood are indisputably perceived as improving opportunities. For example, with respect to the complete ordering $\geq_C$, the family $\mathcal{F}^{\geq_{QO}}$ is the (anti) cumulated lists of outcome $\{k\}$, $\{k-1, k\}$,...,$\{1, 2, ..., k\}$ used to check for first-order stochastic dominance. For any probability distribution $p \in \Delta^{k-1}$, and any $J \in \mathcal{F}^{\geq_{QO}}$, we let $p(J)$ denote the cumulated probability of achieving an outcome in that set. The majorization procedure that we propose as a test for QOEZ dominance between two-group allocations works as follows. For any two such allocations, it first checks whether the symmetric average opportunities are better in one allocation than in the other. If such a dominance is observed, then the allocation with the dominating average is a candidate for a dominating allocation according to Remark 2. To verify that this is indeed so, the test examines, for *each* of the two distributions in the (possibly) dominating allocation, all the mixtures of distributions in the (possibly) dominated allocation that yield the same probability of reaching outcomes in some members of $\mathcal{F}^{\geq_{QO}}$. There may not be any such mixtures, in which case it can be concluded that there is no dominance. If, however, such mixtures exist, then the verdict of dominance is obtained if each of the two distributions in the dominating allocation dominates at least one such mixture of the two distributions in the dominated allocation. The following theorem describes this procedure and shows its equivalence to QOEZ dominance.

Theorem 2 *Suppose that $n = 2$. Let $\mathbf{\Lambda}_i$ (for $i = 1, 2$) be defined by:*

$$\mathbf{\Lambda}_i = \{1\} \cup \{\lambda \in [0,1] : \exists J \in \mathcal{F}^{\geq_{QO}} \ \ s.t.\ q_i(J) = \lambda p_1(J) + (1-\lambda) p_2(J)\}.$$

Then $\mathbf{q} \succsim_Z^{QO} \mathbf{p}$ if and only if $\overline{q} - \overline{p} \in \mathcal{U}_*^{\geq_{QO}}$ and there are $\lambda_i \in \mathbf{\Lambda}_i$ (for $i = 1, 2$) such that $q_1 - (\lambda_1 p_1 + (1-\lambda_1) p_2) \in \mathcal{U}_*^{\geq_{QO}}$ and $q_2 - (\lambda_2 p_1 + (1-\lambda_2) p_2) \in \mathcal{U}_*^{\geq_{QO}}$.

The simplicity of the procedure described by Theorem 2 is illustrated by the following example of two allocations whose dominance relationship is not immediately apparent.

Example 3 *Consider the allocations:*

$\mathbf{p} =$

	1	2	3	4
group 1	$\frac{16}{36}$	$\frac{4}{36}$	$\frac{6}{36}$	$\frac{10}{36}$
group 2	$\frac{13}{36}$	$\frac{3}{36}$	$\frac{12}{36}$	$\frac{8}{36}$

$\mathbf{q} =$

	1	2	3	4
gr 1	$\frac{16}{36}$	$\frac{2}{36}$	$\frac{8}{36}$	$\frac{10}{36}$
gr 2	$\frac{13}{36}$	$\frac{5}{36}$	$\frac{9}{36}$	$\frac{9}{36}$

Observe that $\overline{q} - \overline{p} = \frac{1}{72}(0, 0, -1, 1) \in \mathcal{U}_*^{\geq_C}$. Hence $\mathbf{q}$ is possibly an allocation that dominates allocation $\mathbf{p}$ for the criterion $\succsim_Z^C$. Let us use the procedure described in Theorem 2 to verify that this is indeed the case. Here, $\mathcal{F}^{\geq_C} = \{\{1,2,3,4\}, \{2,3,4\}, \{3,4\}, \{4\}\}$. The sets $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ are therefore respectively *defined as the union of singleton* $\{1\}$ *and the sets of solutions, in the* $[0,1]$ *interval, of the following equations:*

$$10 = 10\lambda_{11} + 8(1-\lambda_{11}) \Rightarrow \lambda_{11} = 1$$
$$18 = 16\lambda_{12} + 20(1-\lambda_{12}) \Rightarrow \lambda_{12} = 1/2$$
$$20 = 20\lambda_{13} + 23(1-\lambda_{13}) \Rightarrow \lambda_{13} = 1$$

for $\mathbf{\Lambda}_1$ *and of the equations:*

$$9 = 10\lambda_{21} + 8(1-\lambda_{21}) \Rightarrow \lambda_{21} = 1/2$$
$$18 = 16\lambda_{22} + 20(1-\lambda_{22}) \Rightarrow \lambda_{22} = 1/2$$
$$23 = 20\lambda_{23} + 23(1-\lambda_{23}) \Rightarrow \lambda_{23} = 0$$

for $\mathbf{\Lambda}_2$*. We thus have* $\mathbf{\Lambda}_1 = \{1/2, 1\}$ *and* $\mathbf{\Lambda}_2 = \{0, 1/2, 1\}$*. Since* $q_1 \succeq^{1st} p_1$*, we have* $q_1 - (\lambda p_1 + (1-\lambda)p_2 \in \mathcal{U}_*^{\geq QO}$ *for* $\lambda = 1 \in \mathbf{\Lambda}_1$*. One can also observe that* $q_2 \succeq^{1st} \frac{1}{2}p_1 + \frac{1}{2}p_2$*. Hence* $\mathbf{q} \succsim_Z^C \mathbf{p}$*.*

Remark 4 *Interestingly, Example 3 also shows that the three elementary operations defined above are not the only ones being considered worth performing by all opportunity inequality averse UEVEU ethical observers. Indeed, it is not possible to go from* $\mathbf{p}$ *to* $\mathbf{q}$ *by a finite sequence of uniform averaging, bilateral equalizing transfers and/or anonymous expected utility improvements. That no equalizing transfers can be performed to go from* $\mathbf{p}$ *to* $\mathbf{q}$ *is clear since neither of the two distributions* p_1 *and* p_2 *first-order stochastically dominates the other. One can also see that no uniform averaging operation, however small, can be performed. Indeed, for any* $\lambda \in [0, 1[$, $q_1 - (\lambda p_1 + (1-\lambda)p_2) \notin \mathcal{U}_*^{\geq C}$*. This is so because the probability of achieving the worst outcome for the first group in allocation* $\mathbf{q}$ *is strictly larger than any mixture of the probabilities of achieving that worst outcome for the two groups in allocation* $\mathbf{p}$ *(*$q_{11} = 16/36 > \lambda 16/36 + (1-\lambda)13/36$ *for all* $0 \leq \lambda < 1$*). Finally, we can show (see Appendix) that there is no margin to perform an anonymous and unanimous utility improvement, however small, on the initial allocation*

$\mathbf{p}$ *in a way that preserves dominance of* $\mathbf{q}$ *over the transformed* $\mathbf{p}$.

We end this section by considering a particular - but theoretically important - case where two of the three types of elementary transformations considered in the preceding subsection coincide with the QOEZ dominance criterion. This case is when the two allocations offer the same average opportunities to the two groups and, therefore, only differ in the way by which this common average opportunity is split between them. In this case, QOEZ dominance actually coincides with the possibility of going from the dominated to the dominating distribution by a finite sequence of equalizing transfers and uniform averaging operations. The following theorem establishes that fact.

Theorem 3 *Suppose that* $n = 2$ *and* $\overline{p} = \overline{q}$. *The three following statements are equivalent:*

1. $\mathbf{q}$ *is obtained from* $\mathbf{p}$ *through uniform averaging or equalizing transfer;*

2. $\mathbf{q} \succsim_{UEVEU}^{QO} \mathbf{p}$*;*

3. $\mathbf{q} \succsim_{Z}^{QO} \mathbf{p}$.

The equivalence established in Theorem 3 provides a simple way to check for dominance in two-group cases between two allocations of the same average opportunities. This is so at least if we focus on the case where outcomes are completely ordered and where, as a result, the dual cone of the set of lists of utility numbers $(u_1, ..., u_k)$ increasing with the outcomes is the set of changes v that generate first-order dominance between distributions. In this case, one can observe the following (as an immediate consequence of Theorem 3 or of Lemma 4 applied to 2 groups).

Remark 5 *Suppose that* $\overline{p} = \overline{q}$ *and* $n = 2$. *Assume that either* $p_1 \succsim^{1st} p_2$ *or* $p_2 \succsim^{1st} p_1$. *Consider the indexing* i_1 *and* i_2 *of the two groups such that* $p_{i_2} \succsim^{1st} p_{i_1}$. *Then* $\mathbf{q} \succsim_{Z}^{C} \mathbf{p}$ *if and only if* $p_{i_2} \succsim^{1st} q_{i_1} \succsim^{1st} p_{i_1}$ *and* $p_{i_2} \succsim^{1st} q_{i_2} \succsim^{1st} p_{i_1}$.

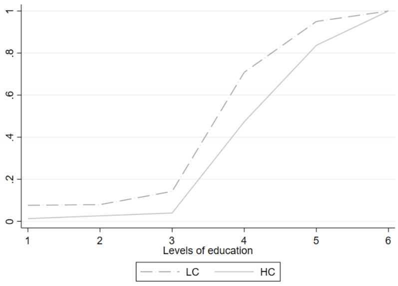
This remark opens the way to a very simple test of opportunity equalization in the two-group case applicable to any two allocations with common average opportunities and where one group is first-order stochastically dominated by the other in one allocation. The test amounts to verifying whether, in the other allocation, the distributions of outcomes of the two groups lie in between those of the two groups in the current allocation in terms of first-order stochastic dominance.

4 Empirical Illustration

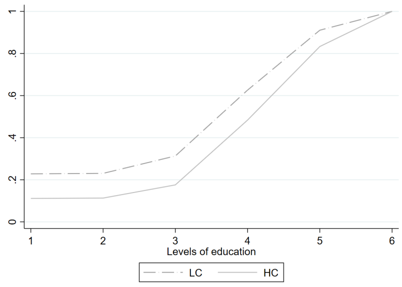
This section applies the QOEZ dominance criterion to the evaluation of allocations of educational opportunities in six major Indian states: Kerala (KR), Maharashtra (MH), West Bengal (WB), Odisha (OD), Andhra Pradesh (AN) and Rajasthan (RJ). As par population census, the first pair of states (KR, MH) have a higher level of literacy than the national average, whereas the last couple of states (AN, RN) are among some of the least literate states of India. WB and OD somewhat fall in between the above two sets. In order to perform the evaluation, we first consider caste as the only (morally irrelevant) characteristic on which the group are formed. We then also introduce gender as an (additional) source of group differentiation.

We base our empirical analysis on the latest available Employment-Unemployment Survey from the NSSO micro-database that covers the survey year of 2011-2012. This survey provides information on the highest level of educational qualification achieved by every member of over 100,000 surveyed households. This is one of the large scale surveys of NSSO that covers almost the entire country with a complex and multi-layered sample design, which grants representativeness at the state level. We however limit our attention to Indian adults aged between 30-40 years who are currently not attending any educational institution as a trainee or student. We consider educational achievement in six levels with illiteracy considered as the worst educational outcome, and having a university degree or above as the best.

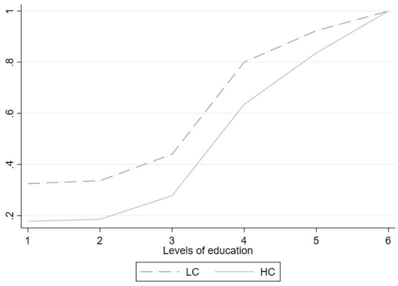
We group adults into caste groups according to the official Indian categories of Scheduled Castes (SC) and Scheduled Tribes (ST), that we merged together as



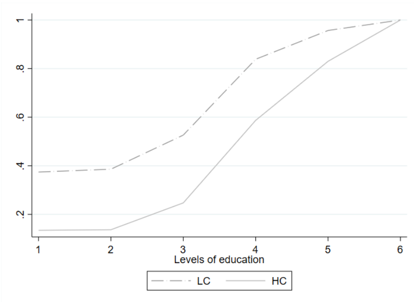
(a) Kerala



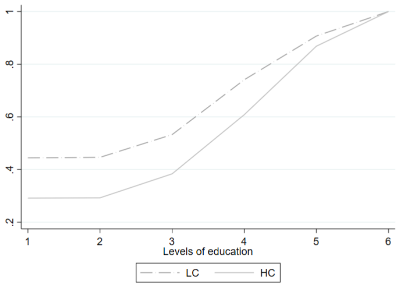
(b) Maharashtra



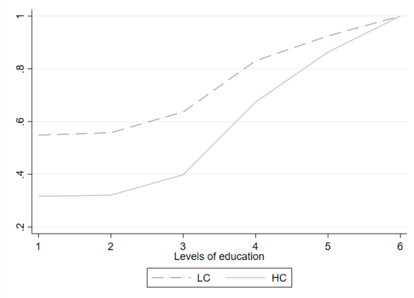
(c) West Bengal



(d) Odisha



(e) Andhra Pradesh



(f) Rajasthan

Figure 2: Education distribution across LC and HC groups: Selected states (2012)

"low castes" (henceforth LC), while the remaining part of the adult population forms what we call the "high castes" (HC). This binary partition of the population into two caste groups is obviously a very rough description of the complexities of the Indian caste system described in details for example in Deshpande (2011). We do not adopt an "individual" interpretation of the group in this empirical application and we rather view the caste themselves as the groups. This means that our criterion is not going to weights the castes by their relative frequency in the population and is going to treat them symmetrically.

It should come as no surprise that, as shown by Figure 2, the distribution of educational outcome in the HC group dominates the distribution of the LC group by first order stochastic dominance in every state. Invoking proposition 2, QOEZ dominance between each pair of states can therefore be tested, first by checking for dominance between LC in the concerned states, and then by checking for dominance between the symmetric average educational distribution over the two caste groups. The conclusion of QOEZ dominance is obtained if each of these two first order stochastic dominance comparisons are obtained in a statistically significant matter.[8] The results of the two stochastic dominance tests are provided in Table 1, along with the overall QOEZ dominance results.

Table 1 reveals that among the 15 pairs of comparisons (between 6 states), 9 concludes on a clear dominance of one state over another by all (educational) opportunity-averse-UEVEU ethical observers. As can be seen, the ranking of the states as per QOEZ dominance more than often mirrors the ranking that could be predicted on the sole basis of their average distribution of educational achievement. Hence the relatively well-educated state of Maharashtra dominates all other states but Kerala while the relatively low-educated state of Rajasthan is dominated by all other states but Odisha. Yet the comparisons reveal some interesting and, possibly, surprising features. One of them is precisely the absence of dominance of Rajasthan by Odisha despite the fact that the latter state has a better average distribution of education opportunities than the former. As it happens, the inequality of educational opportunity in Odisha is so great that it prevents this state from dominating Rajasthan. A similar somewhat counterintuitive - at least when one neglects equality of educational opportunity - absence of dominance is also observed between Kerala - a state often portrayed as exemplary in terms of educational achievement - and Andhra Pradesh. Here again, massive caste-based educational inequality of opportunity in Kerala prevents it from dominating the more equal, but less educated in average, state of Andhra Pradesh.

Besides caste belonging, gender is another important driver of educational

[8]All dominance conclusions in this section are based on a statistical inference methodology proposed in Davidson and Duclos (2000) that uses the union-intersection criterion of Bishop, Formby, and Thistle (1992). For details of the statistical inference methodology see Appendix D of Bennia, Gravel, Magdalou, and Moyes (2022).

		MH	WB	OD	AN	RJ
KR	(LC)	$\nsucc$	$\succ$	$\succ$	$\nsucc$	$\succ$
	(1/2)(LC+HC)	$\nsucc$	$\nsucc$	$\succ$	$\succ$	$\succ$
	Verdict	$\nsucc^C_Z$	$\nsucc^C_Z$	$\succ^C_Z$	$\nsucc^C_Z$	$\succ^C_Z$
MH	(LC)	·	$\succ$	$\succ$	$\succ$	$\succ$
	(1/2)(LC+HC)	·	$\succ$	$\succ$	$\succ$	$\succ$
	Verdict	·	$\succ^C_Z$	$\succ^C_Z$	$\succ^C_Z$	$\succ^C_Z$
WB	(LC)	·	·	$\succ$	$\nsucc$	$\succ$
	(1/2)(LC+HC)	·	·	$\succ$	$\nsucc$	$\succ$
	Verdict	·	·	$\succ^C_Z$	$\nsucc^C_Z$	$\succ^C_Z$
OD	(LC)	·	·	·	$\nsucc$	$\nsucc$
	(1/2)(LC+HC)	·	·	·	$\nsucc$	$\succ$
	Verdict	·	·	·	$\nsucc^C_Z$	$\nsucc^C_Z$
AN	(LC)	·	·	·	·	$\succ$
	(1/2)(LC+HC)	·	·	·	·	$\succ$
	Verdict	·	·	·	·	$\succ^C_Z$

Table 1: Caste dominance: Selected states (2012)[a]

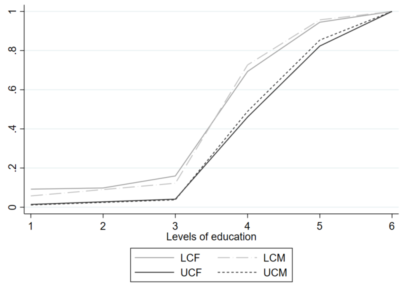
[a] $\succ$ means that the distribution by row first order dominates that by column. $\succ^C_Z$ means row-state dominates column-state by QOEZ criterion (for ordered outcomes). Whereas $\nsucc$ ($\nsucc^C_Z$) means that there is no dominance between the row and the column by first order (by QOEZ). State abbreviations - KR (Kerala), MH (Maharastra), WB (West Bengal), OD (Odisha), AN (Andhra Pradesh), RJ (Rajasthan).

inequality in India. Reducing gender gap in schooling is a major challenge in India as in many other developing areas and is one of the last decade's eight United Nations Millennium Development goals. Several policies have been implemented in India to increase girls' attendance at both primary and secondary school so as to raise women's education levels (Dhaliwal, Duflo, Glennester, and Tulloch, 2013; Muralidharan and Prakash, 2017), with a varying degree of success. It is also unclear how the gender inequality in educational opportunities has evolved among castes. There has been actually very few studies that analyze educational inequality in the caste-gender nexus. Deshpande (2007) finds gender inequality in education to more prominent for the deprived castes. Saha (2013) on the other hand observes that the within-caste distribution of educational expenditure is rather male-skewed for the non-SC/ST. Munshi and Rosenzweig (2006) in case of Mumbai (capital of Maharashtra) finds however that low caste girls obtain better schooling than boys, since boys are often withdrawn from school by their parents and sent to work in traditional low-paid jobs exploiting the intra-caste mutual networks. The salience of the gender gap in educational opportunities is mediated by the caste belonging, which may contribute widening up differences between boys and girls.

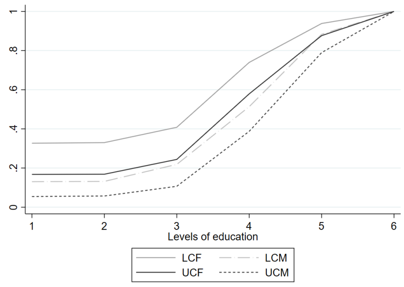
Combining caste and gender leads to the formation of four groups: low caste females (LCF), low caste males (LCM), high caste females (HCF) and high caste males (HCM). The first obvious effect of introducing gender in the analysis is a change in the ubiquitous ordering of the newly formed groups, as shown in Figure 3. While LCF and HCF are always dominated by LCM and HCM, respectively, the caste-gender disentanglement marks the interesting positions of LCM vis-à-vis HCF in different states. In Maharashtra, Andhra and Rajasthan, HCF is dominated by LCM, thereby indicating an acute gender discrimination in these states where educational opportunity is impaired even for HC if they are female. The opposite is true for West Bengal and Odisha, where distribution of education is more unfavorable for LC irrespective of their gender. Interesting is the case of Kerala, the only state (in our analysis) that shows a female dominance among the high caste, although the male-female gap in Kerala is much narrower even for the low castes, in spite of the prominent caste gap there. However, since all four groups can be ordered by first-order dominance in every state (with some rank reversals), we are safe to use Proposition 2 as before to test for QOEZ dominance.

Table 2 shows the results of the QOEZ dominance comparisons in the caste-gender analysis. The table reveals that, except for Maharashtra, the relative order of dominance (between the 15 pairwise comparison) does not change. While Rajasthan retains the status of being the most dominated state when considering a social division by both caste and gender, Maharashtra does not hold up to its dominant position when gender is introduced as an additional source of unfairness along with caste. Maharashtra's poor performance in terms of gender inequality of educational opportunities prevents in effect this state from dominating West Bengal and Odisha, while the neglects of gender was leading to such domination.

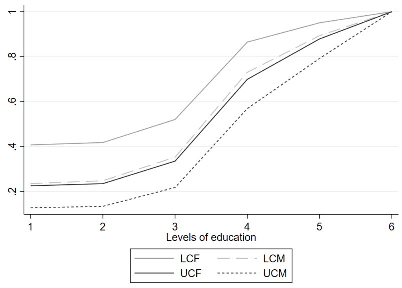
Table 2 provides some further clarification of the Kerala-Andhra dominance



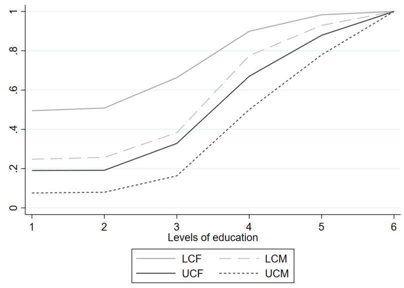
(a) **KR**: $LCF < LCM < HCM < HCF$



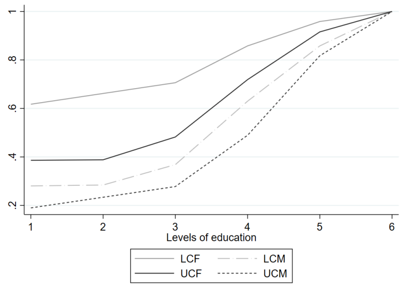
(b) **MH**: $LCF < HCF < LCM < HCM$



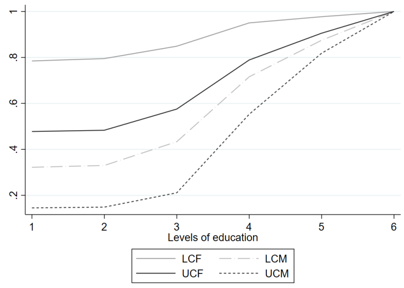
(c) **WB**: $LCF < LCM < HCF < HCM$



(d) **OD**: $LCF < LCM < HCF < HCM$



(e) **AN**: $LCF < HCF < LCM < HCM$



(f) **RJ**: $LCF < HCF < LCM < HCM$

Figure 3: Education distribution across caste and gender: Selected states (2012)[a]

[a]LCF: low caste females, LCM: low caste males, HCF: high caste females HCM: high caste males. State abbreviations - KR (Kerala), MH (Maharastra), WB (West Bengal), OD (Odisha), AN (Andhra Pradesh), RJ (Rajsthan).

failure as well. While Kerala fails to dominate Andhra both in the caste and the caste-gender analysis of educational opportunities, the latter analysis reveals that the break in this dominance comes from comparing the symmetric average distribution over four groups. Kerala holds dominance over Andhra while comparing the worst groups (LCF in both states) and also the two worst groups (that is, all LC in Kerala and all females in Andhra). This dominance survives the addition of the third worst group too, that is HCM in Kerala and LCM in Andhra. But in spite of the favorable position of HCF within Kerala (notice it is the best group

		MH	WB	OD	AN	RJ
KR	`(worst type)`	$\succ$	$\succ$	$\succ$	$\succ$	$\succ$
	(1/2)`(two worst types)`	$\succ$	$\nsucc$	$\succ$	$\succ$	$\succ$
	(1/3)`(three worst types)`	$\nsucc$	$\cdot$	$\succ$	$\succ$	$\succ$
	(1/4)`(all four types)`	$\cdot$	$\cdot$	$\succ$	$\nsucc$	$\succ$
	Verdict	$\nsucc^C_Z$	$\nsucc^C_Z$	$\succ^C_Z$	$\nsucc^C_Z$	$\succ^C_Z$
MH	`(worst type)`	-	$\succ$	$\succ$	$\succ$	$\succ$
	(1/2)`(two worst types)`	-	$\nsucc$	$\nsucc$	$\succ$	$\succ$
	(1/3)`(three worst types)`	-	$\cdot$	$\cdot$	$\succ$	$\succ$
	(1/4)`(all four types)`	-	$\cdot$	$\cdot$	$\succ$	$\succ$
	Verdict	-	$\nsucc^C_Z$	$\nsucc^C_Z$	$\succ^C_Z$	$\succ^C_Z$
WB	`(worst type)`	-	-	$\succ$	$\succ$	$\succ$
	(1/2)`(two worst types)`	-	-	$\succ$	$\succ$	$\succ$
	(1/3)`(three worst types)`	-	-	$\succ$	$\nsucc$	$\succ$
	(1/4)`(all four types)`	-	-	$\succ$	$\cdot$	$\succ$
	Verdict	-	-	$\succ^C_Z$	$\nsucc^C_Z$	$\succ^C_Z$
OD	`(worst type)`	-	-	-	$\nsucc$	$\succ$
	(1/2)`(two worst types)`	-	-	-	$\cdot$	$\succ$
	(1/3)`(three worst types)`	-	-	-	$\cdot$	$\nsucc$
	(1/4)`(all four types)`	-	-	-	$\cdot$	$\cdot$
	Verdict	-	-	-	$\nsucc^C_Z$	$\nsucc^C_Z$
AN	`(worst type)`	-	-	-	-	$\succ$
	(1/2)`(two worst types)`	-	-	-	-	$\succ$
	(1/3)`(three worst types)`	-	-	-	-	$\succ$
	(1/4)`(all four types)`	-	-	-	-	$\succ$
	Verdict	-	-	-	-	$\succ^C_Z$

Table 2: Caste and gender dominance: Selected states (2012)[a]

[a] $\succ$ ($\nsucc$) means that the distribution by row first order dominates (does not dominate) that by column. $\succ^C_Z$ ($\nsucc^C_Z$) means row-state dominates (does not dominate) column-state by QOEZ criterion with ordered realizations. Notice that we can conclude on non-dominance by QOEZ criterion if dominance fails for any of the sequential summation of types and there is no need to test dominance for others, in which case the corresponding cell is marked by a dot ($\cdot$). State abbreviations - KR (Kerala), MH (Maharastra), WB (West Bengal), OD (Odisha), AN (Andhra Pradesh), RJ (Rajsthan).

there), the domination of Kerala (over Andhra) no longer holds because of the relatively favorable educational distribution of Andhra HCM over Keralite HCF. Hence, the relatively good performance of Kerala in allocating educational opportunities among gender is not sufficient to offset the poor performance of this state in terms of caste-based inequality (that was already noticed by Deshpande (2000) some time ago using different tools).

5 Conclusion

This paper provides a robust and operational definition of what it means for one inter-group allocation of opportunities - defined as probabilities of achieving outcomes of interest - to be better than another for a (reasonably) large spectrum of ethical points of view. Our operational definition, which is shown to coincide with the unanimity of all rankings emanating from inequality-of opportunity-averse UEVEU ethical observers, is the test of extended zonotope inclusion. The zonotope set of an allocation of opportunities is the convex hull of all partial sums, over groups, of their probability distributions. The extended zonotope of an allocation of opportunities is simply its zonotope set translated by transformations of the distributions of outcomes that are considered worthwhile by some *a priori* ranking of the outcomes. According to this criterion therefore, undisputable improvements in allocations of opportunities are associated with a shrinking - in the sense of set inclusion - of the associated extended zonotope set. We also show how this Zonotope inclusion test can be made extremely simple in many cases of interest. Among the cases considered are those where there are only two groups between which opportunities are allocated, and those where the number of groups is arbitrary, but where the groups can all be ordered in terms of the expected utility associated with the probability distribution faced by their members. However we also provide an example of a situation where extended zonotope inclusion may be difficult to verify. We also identify elementary transformations of the allocation of opportunities that are considered worthwhile under the extended zonotope inclusion criterion and, in the specific case of two groups facing a given average distribution of outcomes, we identify them exactly. Last, but not least, the paper illustrates the usefulness of the criterion by applying it to compare gender- and caste-based allocation of educational opportunities in a few Indian states. The empirical analysis in particular emphasizes the importance of caste and gender inequality when assessing those states' differing performance in providing opportunities for education to their inhabitants. The analysis reveals, among other things, that the good average educational performance of Kerala hides major between-caste inequalities of opportunity that prevent this state from dominating many others. In the same vein, the analysis also points to significant gender inequalities of educational opportunity in Maharashtra that also prevent this wealthy and well-educated state from dominating others.

These findings open many future research perspectives. One of them is further empirical analysis. As shown in this paper, the extended zonotope inclusion criterion is a test that can easily be implemented in many empirical cases. It would therefore be interesting to apply this test to other contexts requiring an inequality-sensitive assessment of allocations of opportunity. Another closely related avenue of research would the axiomatic identification of numerical indices that could supplement the incomplete comparisons of allocation of opportunities provided by the extended zonotope inclusion criterion, while remaining compatible with it. From a theoretical view point, it would also be useful to identify precisely which elementary transformations of the allocations of opportunities lie behind extended zonotope inclusion. While these elementary transformations have been identified in the simple case of two groups who share the same symmetric average distributions over outcomes, they do not suffice to characterize extended zonotope inclusion when there are more than two groups and/or when average distribution over outcomes differs. Another line of inquiry that would be worth pursuing is the identification of simple finite procedures for verifying extended zonotope inclusion that apply to all logically conceivable cases. While Example 2 suggests that this may be difficult, we believe it is worth another try. Last, but certainly not least, it might also be worth going beyond the *ex ante* standpoint, favored in this paper, of appraising opportunities *before* they are realized. When outcomes are totally ordered, the only clear indication of improvement to a group's opportunities is first-order stochastic dominance. Yet this criterion is not sensitive to mean preserving spreads or Pigou-Dalton transfers of outcomes. Defining these mean preserving spreads requires, of course, that outcomes be measured in a cardinally meaningful way. It would be worth exploring an extension of the criterion characterized in this paper that would coincide with a more restricted unanimity of all opportunity-inequality-averse UEVEU ethical observers who assume that the utility numbers used to evaluate expected utility are both increasing and concave with respect to outcomes.

References

Ahn, D. S. (2008): "Ambiguity without a State Space," *Review of Economic Studies*, 75, 3–28.

Andreoli, F., T. Havne, and A. Lefranc (2019): "Robust Inequality of Opportunities Comparisons: Theory and Application to Early Childhood Policy Evaluation," *Review of Economics and Statistics*, 101, 355–369.

Atkinson, A. B. (1981): "The Measurement of Economic Mobility," in *Inkomensverdeling en Openbare Financien*, ed. by P. Eggelshoven, and L. van Gemerden. Het Spectrum, Leiden.

Bénabou, R., and E. A. Ok (2001): "Mobility as Progressivity:Ranking income process according to Equality of Opportunity," NBER working paper 8431.

Bennia, F., N. Gravel, B. Magdalou, and P. Moyes (2022): "Comparing Distributions of Body Mass Index Categories," *The Scandinavian Journal of Economics*, 124, 69–103.

Berge, C. (1959): *Espaces topologiques et fonctions multivoques*. Dunod, Paris.

Bishop, C. M., J. P. Formby, and P. Thistle (1992): "Convergence of the South and the Non-South Income Distributions, 1969-1979," *American Economic Review*, 82, 262–272.

Brunori, P., F. Ferreira, and V. Peragine (2021): "Prioritarism and Equality of Opportunity," in *Prioritarism in Practice*, ed. by M. Adler, and O. Norheim. Cambridge University Press, Cambridge, UK.

Dardanoni, V. (1993): "Measuring Social Mobility," *Journal of Economic Theory*, 61, 372–394.

Davidson, R., and J. Y. Duclos (2000): "Statistical Inference for Stochastic Dominance and for the Measurement of Poverty and Inequality," *Econometrica*, 58, p.1435–1465.

Deshpande, A. (2000): "Does caste still define disparity ? A look at inequality in Kerala, India," *American Economic Review*, 90, 322–325.

——— (2007): "Overlapping identities under liberalization: gender and caste in India," *Economic Development and Cultural Change*, 55, 735–760.

——— (2011): *The Grammar of Castes: Economic Discriminations in Contemporary India*. Oxford University Press, Oxford,UK.

Dhaliwal, I., E. Duflo, R. Glennester, and C. Tulloch (2013): "Comparative Cost-Effectiveness Analysis to Inform Policy in Developing Countries: A General Framework with Applications for Education," in *Education Policies in Developping Countries*, ed. by P. Glewwe, pp. 285–338. University of Chicago Press, Chicago.

Donaldson, D., and J. A. Weymark (1998): "A Quasi-Ordering is the Intersection of Orderings," *Journal of Economic Theory*, 78, 382–387.

Dworkin, R. (1981): "What is Equality ? Part 1: Equality of Welfare, Part 2: Equality of Ressources," *Philosophy and Public Affairs*, 10, 185–246.

Ferreira, F. H. G., and J. Gignoux (2011): "The Measurement of Equality of Opportunities: Theory and an Application to Latin America," *The Review of Income and Wealth*, 57, 622–657.

Fleurbaey, M. (2008): *Fairness, responsibility, and welfare*. OUP Oxford.

Fleurbaey, M. (2010): "Assessing Risky Social Situations," *Journal of Political Economy*, 118, 649–680.

——— (2018): "Welfare Economics, Risk and Uncertainty," *Canadian Journal of Economics*, 51, 5–40.

Gravel, N., T. Marchant, and A. Sen (2011): "Comparing Societies with Different Numbers of Individuals on the Basis of their Average Advantage," in *Social Ethics and Normative Econonomics: Essays in Honour of Serge-Christophe Kolm*, ed. by M. Fleurbaey, M. Salles, and J. A. Weymark, pp. 261–277. Springer Verlag.

——— (2012): "Uniform Utility Criteria for Decision Making under Ignorance or Objective Ambiguity," *Journal of Mathematical Psychology*, 56, 297–315.

Kolm, S. C. (1977): "Multidimensional Egalitarianisms," *Quarterly Journal of Economics*, 91, 1–13.

Koshevoy, G. (1995): "Multivariate Lorenz majorization," *Social Choice and Welfare*, 12, 93–102.

——— (1998): "The Lorenz zonotope and multivariate majorizations," *Social Choice and Welfare*, 15, 1–14.

Koshevoy, G., and K. Mosler (1996): "The Lorenz zonoid of a multivariate distribution," *Journal of the American Statistical Association*, 91, 873–882.

——— (2007): "Multivariate Dominance Basded on Zonoids," *AStA Advances in Statistical Analysis*, 91, 57–76.

LEFRANC, A., N. PISTOLESI, AND A. TRANNOY (2009): "Equality of Opportunity and Luck: Definition and Testable Condition with an application to income in France," *Journal of Public Economics*, 93, 1189–1207.

LEFRANC, A., AND A. TRANNOY (2017): "Equality of opportunity, moral hazard and the timing of luck," *Social Choice and Welfare*, 49, 469–497.

MARIOTTI, M., AND R. VENEZIANI (2017): "Opportunities as Chances: Maximizing the probability that everybody succeeds," *Economic Journal*, 128, 1609–1633.

MARTINEZ, M., E. SCHOCKKAERT, AND D. VANDEGAER (2001): "Three Meanings of Intergenerational Mobility," *Economica*, 68, 519–537.

MUNSHI, K., AND M. ROSENZWEIG (2006): "Traditional Institutions meet the Modern World: Caste, Gender and Schooling Choice in a globalized economy," *American Economic Review*, 96, 1225–1252.

MURALIDHARAN, K., AND N. PRAKASH (2017): "Cycling to School: Increasing Secondary School Enrollment for Girls in India," *American Economic Journal: Applied Economics*, 9, 321–350.

NEUMANN, J. V., AND O. MORGENSTERN (1947): *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, NJ, USA.

OOGHE, E., E. SCHOKKAERT, ET AL. (2007): "Equality of opportunity versus equality of opportunity sets," *Social Choice and Welfare*, 28(2), 209–230.

PERAGINE, V. (2004): "Ranking income distributions according to equality of opportunity," *Journal of Economic Inequality*, 2, 11–30.

ROCKAFELLAR, R. T. (1970): *Convex Analysis*. Princeton University Press, Princeton, NJ, USA.

ROEMER, J. E. (1996): *Theories of Justice*. Harvard University Press, Cambridge MA.

ROEMER, J. E., AND A. TRANNOY (2016): "Equality of Opportunity: Theory and Measurement," *Journal of Economic Literature*, 54, 1288–1332.

SAHA, A. (2013): "An Assessment of Gender Discrimination in Schooling Expenditures on Education in India," *Oxford Development Studies*, 41, 220–238.

SHORROCKS, A. F. (1983): "Ranking Income Distributions," *Economica*, 50, 3–17.

——— (1984): "Inequality Decomposition by Population Subgroups," *Econometrica*, 52, 1369–1386.

VALLENTYNE, P. (2002): "Brute Luck, Option Luck and Equality of Initial Opportunities," *Ethics*, 112, 529–557.

VAN DE GAER, D. (1993): "Equality of opportunity and investment in human capital," Ph.D. thesis, KUL, Leuven.

A Notations and Proof of the main results

A.1 Mathematical Notation

The (possibly) non standard mathematical notations and definitions used in this paper are as follows. The $k-1$ dimensional simplex is denoted by Δ^{k-1} and is defined by

$$\Delta^{k-1} = \left\{ (p_1, ..., p_k) \in [0,1]^k : \sum_{h=1}^{k} p_h = 1 \right\}$$

For any two distributions p and $q \in \Delta^{k-1}$, we say that q dominates p with respect to the first-order stochastic dominance, denoted $q \succsim^{1st} p$, if and only if $\sum_{h=j}^{k} q_h \geq \sum_{h=j}^{k} p_h$ for any $j = 1, ..., k$.

The convex hull of a collection of vectors $\{v^1, ..., v^n\}$ in $\mathbb{R}^k$ is defined by

$$Co\{v^1, ..., v^n\} = \left\{ x \in \mathbb{R}^k : \ x = \sum_{i=1}^{n} \lambda_i v^i \text{ for some } (\lambda_1, ..., \lambda_n) \in \Delta^{n-1} \right\}.$$

A matrix $\mathbf{m} \in \mathbb{R}_+^{nk}$ is row-stochastic if it satisfies $\sum_{h=1}^{k} m_{ih} = 1$ for all i, and is bistochastic if it is row-stochastic and satisfies $\sum_{i=1}^{k} m_{ih} = 1$ for every h.

Given two vectors u and $v \in \mathbb{R}^k$, we say that u weakly Lorenz dominates v if the inequality $\sum_{g=1}^{h} u_{(g)} \geq \sum_{g=1}^{h} v_{(g)}$ holds for all $h = 1, ..., k$, where $(v_{(g)})_g$ is the increasingly ordered vector associated to v.

By a binary relation $\succsim$ on a set $\mathbf{\Omega}$, we mean a subset of $\mathbf{\Omega} \times \mathbf{\Omega}$. Following the convention in economics, we write $x \succsim y$ instead of $(x, y) \in \succsim$. Given a binary relation $\succsim$, we define its symmetric factor $\sim$ by $x \sim y \iff x \succsim y$ and $y \succsim x$ and its asymmetric factor $\succ$ by $x \succ y \iff x \succsim y$ and not $(y \succsim x)$. A binary relation $\succsim$ on $\mathbf{\Omega}$ is reflexive if the statement $x \succsim x$ holds for every x in $\mathbf{\Omega}$, is transitive if $x \succsim z$ always follows $x \succsim y$ and $y \succsim z$ for any $x, y, z \in \mathbf{\Omega}$, is complete if $x \succsim y$ or $y \succsim x$ holds for every distinct x and y in $\mathbf{\Omega}$ and is antisymmetric if $x \succsim y$ and $y \succsim x$ implies $x = y$ for any two x and y in $\mathbf{\Omega}$. A reflexive, transitive and complete binary relation is called an ordering and a reflexive and transitive binary relation is called a quasi-ordering.

A.2 Proofs

A.2.1 Theorem 1

We first show that Statement 2 of Theorem 1 implies Statement 1 of that theorem and, therefore, that $\mathbf{Z}(\mathbf{q}) + \mathcal{U}_*^{\geq QO} \subseteq \mathbf{Z}(\mathbf{p}) + \mathcal{U}_*^{\geq QO}$. Since $\mathcal{U}_*^{\geq QO}$ is a cone, it amounts to showing that $\mathbf{Z}(\mathbf{q}) \subseteq \mathbf{Z}(\mathbf{p}) + \mathcal{U}_*^{\geq QO}$. By Lemma 1, it is sufficient to show that, for any $\alpha_1, ..., \alpha_n \in \{0,1\}^n$, there exists $v \in \mathcal{U}_*^{\geq QO}$ and $\theta_1, ..., \theta_n \in [0,1]^n$ such that:

$$\sum_{i=1}^{n} \alpha_i q_i = \sum_{i=1}^{n} \theta_i p_i + v. \tag{8}$$

Note that since $\sum_{j=1}^{k} v_j = 0$ (by Remark 1) and p_i and q_i both belong to Δ^{k-1}, we necessarily have $\sum_{i=1}^{n} \theta_i = m$, where $m = Card\{i : \alpha_i = 1\}$. Hence, by re-indexing the distributions q_i (for $i = 1, ..., n$) in such a way that $\alpha_i = 1$ for $i = 1, ..., m$, Expression (8) can be equivalently written as: $\frac{1}{m} \sum_{i=1}^{m} q_i = \sum_{i=1}^{n} \frac{\theta_i}{m} p_i + \frac{1}{m} v$. Define $\mathbf{D} := \overline{q} - Co\{p_1, ..., p_n\}$. We need to show that $\mathbf{D} \cap \mathcal{U}_*^{\geq QO} \neq \emptyset$. Suppose by contradiction that $\mathbf{D} \cap \mathcal{U}_*^{\geq QO} = \emptyset$. Since $\mathbf{D}$ is a convex polytope[9] and $\mathcal{U}_*^{\geq QO}$ is a closed convex cone, one can conclude from Theorem 2 at p. 80 of Berge (1959) that there are vectors $(d_1^*, ...d_k^*) \in \mathbf{D}$ and $(v_1^*, ...v_k^*) \in \mathcal{U}_*^{\geq QO}$ such that

$$\left(\sum_{h=1}^{k} (d_h^* - v_h^*)^2 \right)^{1/2} = \min_{(d_1, ..., d_k) \in D, (v_1, ..., v_k) \in \mathcal{U}_*^{\geq QO}} \left(\sum_{h=1}^{k} (d_h - v_h)^2 \right)^{1/2}$$

by continuity of the Euclidian norm, and using the fact that the set $\mathbf{D} \times \mathcal{U}_*^{\geq QO}$ on which it is minimized can be made compact by taking a suitable intersection of $\mathcal{U}_*^{\geq QO}$ with some closed ball in $\mathbb{R}^k$. Define the vector $(\widehat{v}_1, ..., \widehat{v}_k)$ by $\widehat{v}_h = v_h^* - d_h^*$ for $h = 1, ..., k$. Then the hyperplane passing through $(v_1^*, ...v_k^*)$ and orthogonal to $(\widehat{v}_1, ..., \widehat{v}_k)$ strongly separates $\mathbf{D}$ and $\mathcal{U}_*^{\geq QO}$ in the sense that

$$\inf_{(v_1, ..., v_k) \in \mathcal{U}_*^{\geq QO}} \sum_{h=1}^{k} v_h \widehat{v}_h \geq \sum_{h=1}^{k} v_h^* \widehat{v}_h > \sup_{(d_1, ..., d_k) \in D} \sum_{h=1}^{k} d_h \widehat{v}_h \tag{9}$$

[9] see Rockafellar (1970), p. 12. A convex polytope is the convex hull of a finite family of points, called the *vertices* or *extreme points* of this set.

Since $(0,...,0) \in \mathcal{U}_*^{\geq QO}$, one must have that $0 \geq \sum_{h=1}^k v_h^* \widehat{v}_h$. Also, since $\lambda(v_1^*,...,v_k^*) \in \mathcal{U}_*^{\geq QO}$ $\forall \lambda > 0$, one must also have that $\sum_{h=1}^k v_h^* \widehat{v}_h \geq 0$. Indeed, assuming $\sum_{h=1}^k v_h^* \widehat{v}_h < 0$ would be contradictory, after taking a suitably large λ, with the strict inequality (9). These two last inequalities enable therefore one to rewrite inequality (9) more precisely as:

$$\inf_{(v_1,...,v_k) \in \mathcal{U}_*^{\geq QO}} \sum_{h=1}^k v_h \widehat{v}_h \geq 0 > \sup_{(d_1,...,d_k) \in D} \sum_{h=1}^k d_h \widehat{v}_h \tag{10}$$

By the first of these two inequalities, we conclude that $(\widehat{v}_1,...,\widehat{v}_k)$ belongs to the dual cone of the set $\mathcal{U}_*^{\geq QO}$, which is itself the dual cone of the set $\mathcal{U}^{\geq QO}$. By the bipolar theorem for convex cones (see for example Theorem 14.1 in Rockafellar (1970)), it therefore follows that the dual cone of $\mathcal{U}_*^{\geq QO}$is $\mathcal{U}^{\geq QO}$ so that $(\widehat{v}_1,...,\widehat{v}_k) \in \mathcal{U}^{\geq QO}$. Now since Statement 2 of the theorem holds, we know that the inequality

$$\sum_{i=1}^n \Phi\left(\sum_{h=1}^k q_{ih} u_h\right) \geq \sum_{i=1}^n \Phi\left(\sum_{h=1}^k p_{ih} u_h\right)$$

holds for all concave Φ and all lists of real numbers $(u_1,...,u_k) \in \mathcal{U}^{\geq QO}$. By the Hardy-Littlewood-Polya theorem (see for example Berge (1959), p. 191), this is equivalent to the requirement that the list of n numbers $\left(\sum_{h=1}^k q_{1h} u_h, ..., \sum_{h=1}^k q_{nh} u_h\right)$ Lorenz dominates the list of $n$numbers $\left(\sum_{h=1}^k p_{1h} u_h, ..., \sum_{h=1}^k p_{nh} u_h\right)$, for all $(u_1,...,u_k) \in \mathcal{U}^{\geq QO}$. In particular this is true for $(\hat{u}_1,...,\hat{u}_k)$, and thus there exists an indexing $i_1(\widehat{u}),...,i_n(\widehat{u})$ such that:[10]

$$\sum_{h=1}^k p_{i_1(\widehat{u})h} \widehat{u}_h \leq \sum_{h=1}^k p_{i_2(\widehat{u})h} \widehat{u}_h \leq ... \leq \sum_{h=1}^k p_{i_n(\widehat{u})h} \widehat{u}_h$$

and:

$$\sum_{i=1}^m \sum_{h=1}^k q_{ih} \widehat{u}_h \geq \sum_{j=1}^m \sum_{h=1}^k p_{i_j(\widehat{u})h} \widehat{u}_h. \tag{11}$$

However, by the second inequality of Expression (10),we have (remembering the definition of $\mathbf{D}$):

$$0 > \sum_{h=1}^k \overline{q}_h \widehat{u}_h - \sum_{h=1}^k p_{ih} \widehat{u}_h \tag{12}$$

[10](which depends of course upon the k-tuple $(\widehat{u}_1,...,\widehat{u}_k)$)

for all $i = 1, ..., n$. It follows therefore from Inequalities (11) and (12) that:

$$\sum_{h=1}^{k} p_{ih}\widehat{u}_h > \sum_{h=1}^{k} \overline{q}_h \widehat{u}_h \geq \frac{1}{m}\sum_{j=1}^{m}\sum_{h=1}^{k} p_{i_j(\widehat{u})h}\widehat{u}_h, \text{ for } i = 1, ..., n,$$

which is not possible. This concludes the proof of the first implication.

Let us now prove the reverse implication. Suppose that Statement 1 of the Theorem holds and pick any $(u_1, ..., u_k) \in \mathcal{U}^{\geq QO}$. We must show, using again the Hardy-Littlewood-Polya theorem, that the list of n numbers $\left(\sum_{h=1}^{k} q_{1h}u_h, ..., \sum_{h=1}^{k} q_{nh}u_h\right)$ Lorenz dominates $\left(\sum_{h=1}^{k} p_{1h}u_h, ..., \sum_{h=1}^{k} p_{nh}u_h\right)$. Without loss of generality (since the ranking of allocations is anonymous), we can write the indices of the rows of the two matrices $\mathbf{q}$ and $\mathbf{p}$ in such a way that the two lists are increasingly ordered so that:

$$\sum_{h=1}^{k} q_{1h}u_h \leq ... \leq \sum_{h=1}^{k} q_{nh}u_h \text{ and} \sum_{h=1}^{k} p_{1h}u_h \leq \sum_{h=1}^{k} p_{nh}u_h.$$

Hence, we need to show that for any $n_0 \leq n - 1$,

$$\sum_{i=1}^{n_0}\sum_{h=1}^{k} p_{1h}u_h \leq \sum_{i=1}^{n_0}\sum_{h=1}^{k} q_{1h}u_h$$

Since statement 1 of the theorem holds, there exists $v \in \mathcal{U}_*^{\geq QO}$ and $\theta_1, ..., \theta_n \in [0, 1]$ such that $\sum_{i=1}^{n} \theta_i = n_0 \leq n$, and $\sum_{i=1}^{n_0} q_i = \sum_{l=1}^{n} \theta_l p_l + v$ It thus follows that:

$$\begin{aligned}
\sum_{i=1}^{n_0}\sum_{h=1}^{k} q_{1h}u_h &= \sum_{j=1}^{n} \theta_h \sum_{h=1}^{k} p_{jh}u_h + \sum_{h=1}^{k} v_h u_h \geq \sum_{j=1}^{n} \theta_h \sum_{h=1}^{k} p_{jh}u_h \text{ (since } v \in \mathcal{U}_*^{\geq QO}) \\
&\geq \sum_{j=1}^{n_0} \theta_j \sum_{h=1}^{k} p_{jh}u_h + \sum_{g=n_0+1}^{n} \theta_g \sum_{h=1}^{k} p_{n_0 h}u_h \text{ (rows are ordered)} \\
&= \sum_{j=1}^{n_0} \theta_j \sum_{h=1}^{k} p_{jh}u_h + \sum_{j=1}^{n_0}[1 - \theta_j] \sum_{h=1}^{k} p_{n_0 h}u_h \text{ (since } \sum_{j=1}^{n} \theta_i = n_0) \\
&\geq \sum_{j=1}^{n_0} \theta_j \sum_{h=1}^{k} p_{jh}u_h,
\end{aligned}$$

as required.

A.2.2 Theorem 2

Using the reasoning in the proof of Remark 2, one can observe that if $n = 2$, the statement $\mathbf{q} \succsim_Z^{QO} \mathbf{p}$ is equivalent to the requirement that $\overline{q} - \overline{p} \in \mathcal{U}_*^{\geq QO}$ and that there exist θ_1 and $\theta_2 \in [0,1]$ such that $q_1 - (\theta_1 p_1 + (1-\theta_1)p_2) \in \mathcal{U}_*^{\geq QO}$ and $q_2 - (\theta_2 p_1 + (1-\theta_2)p_2)$. Since these θ_1 and θ_2 may belong respectively to $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$, this establishes the direct implication.

Proving the other implication amounts to showing that the statement $\mathbf{q} \succsim_Z^{QO} \mathbf{p}$ implies the existence of $\lambda_1 \in \mathbf{\Lambda}_1$ such that $q_1 - (\lambda_1 p_1 + (1-\lambda_1)p_2) \in \mathcal{U}_*^{\geq QO}$ (the argument being similar for λ_2). If $q_1 - p_1 \in \mathcal{U}_*^{\geq QO}$, then one selects $\lambda_1 = 1 \in \mathbf{\Lambda}_1$ and the proof is over. If $q_1 - p_1 \notin \mathcal{U}_*^{\geq QO}$, then we know that since $q_1 - (\theta_1 p_1 + (1-\theta_1)p_2) \in \mathcal{U}_*^{\geq QO}$ for some $\theta_1 \in [0,1]$, there exists some $v_1 \in \mathcal{U}_*^{\geq QO}$ such that $q_1 = \theta_1 p_1 + (1-\theta_1)p_2 + v_1$ Let $\mathbf{D}(q_1)$ denote the (compact) set of distributions of opportunities that are weakly dominated by q_1, with respect to the quasi-ordering, defined by:

$$\mathbf{D}(q_1) = \{x \in \Delta^{k-1} : q_1 - x \in \mathcal{U}_*^{\geq QO}\}$$

Consider the continuous map $x : [0,1] \to [0,1]$ defined by $x(t) = tp_1 + (1-t)p_2$. Since $q_1 - p_1 \notin \mathcal{U}_*^{\geq QO}$ one has that $x(1) \notin \mathbf{D}(q_1)$ while $x(\theta_1) \in \mathbf{D}(q_1)$. Let $\overline{\theta}_1$be defined by:

$$\overline{\theta}_1 = \max\{t \geq \theta_1 : x(t) \in \mathbf{D}(q_1)\} \tag{13}$$

We then have $\overline{\theta}_1 \in [\theta_1, 1[$ and $x(\overline{\theta}_1) \in \mathbf{D}(q_1)$. We therefore have:

$$q_1 = \overline{\theta}_1 p_1 + (1-\overline{\theta}_1)p_2 + \overline{v}_1$$

for some $\overline{v}_1 \in \mathcal{U}_*^{\geq QO}$. Also observe that $\overline{v}_1$ must be such that $\sum_{j \in J} \overline{v}_{1j} = 0$ for some $J \in \mathcal{F}^{\geq QO}$. Indeed, using Expression (7), assuming that $\sum_{j \in J} \overline{v}_{1j} > 0$ for all $J \in \mathcal{F}^{\geq QO}$ would imply the possibility of increasing a bit the t above $\overline{\theta}_1$ while maintaining $x(t)$ in the set $\mathbf{D}(q_1)$ in the maximization described by Expression (13), and will therefore be contradictory. Hence for the set J where $\sum_{j \in J} \overline{v}_{1j} = 0$, one has $q_1(J) = \overline{\theta}_1 p_1(J) + (1 - \overline{\theta}_1 p_2(J)$ and this completes the proof.

A.2.3 Theorem 3

The fact that Statement 1 implies Statement 2 has been proved (for any number of groups) by Lemma 3 while the implication of Statement 3 by Statement 1 has been established by Theorem 1. We therefore only need to prove that Statement 3 implies Statement 1. Suppose therefore that $\mathbf{q} \succsim_Z^{QO} \mathbf{p}$

• Consider first the case where $p_2 - p_1 \in \mathcal{U}_*^{\geq QO}$.[11] Then $\mathbf{Z}(\mathbf{p}) + \mathcal{U}_*^{\geq QO} \subseteq \{\theta p_1 + v : \theta \in [0, 2],\ v \in \mathcal{U}_*^{\geq QO}\}$. Since $\mathbf{q} \succsim_Z^{QO} \mathbf{p}$ we have $q_1 = \theta_1 p_1 + v_1;\ q_2 = \theta_2 p_1 + v_2$, where $\theta_1, \theta_2 \in [0, 2]$ and $v_1, v_2 \in \mathcal{U}_*^{\geq QO}$. Now, q_1 and q_2 being both in Δ^{k-1} and v_1 and v_2 having both their components summing to zero, we must have $\theta_1 = \theta_2 = 1$. As a result $q_1 = p_1 + v_1$ and $q_2 = p_1 + v_2$. Since $p_1 + p_2 = q_1 + q_2$, we have $p_2 = p_1 + v_1 + v_2$. Hence $q_1 = p_1 + v_1$, $q_2 = p_2 - v_1$ and $p_2 - p_1 - v_1 = v_2 \in \mathcal{U}_*^{\geq QO}$, which means that $\mathbf{q}$ has been obtained from $\mathbf{p}$ through an equalizing transfer.

• Consider now the case where neither $p_2 - p_1 \in \mathcal{U}_*^{\geq QO}$ nor $p_1 - p_2 \in \mathcal{U}_*^{\geq QO}$. Since $\mathbf{Z}(\mathbf{q}) + \mathcal{U}_*^{\geq QO} \subseteq \mathbf{Z}(\mathbf{p}) + \mathcal{U}_*^{\geq QO}$ and both q_1 and $q_2 \in \mathbf{Z}(\mathbf{q}) + \mathcal{U}_*^{\geq QO}$, there are numbers θ_1^1, θ_1^1, θ_1^2 and $\theta_2^2 \in [0, 1]$ satisfying $\theta_1^1 + \theta_2^1 = \theta_1^2 + \theta_2^2 = 1$ such that $q_1 = \theta_1^1 p_1 + \theta_2^1 p_2 + v_1$ and $q_2 = \theta_1^2 p_1 + \theta_2^2 p_2 + v_2$, for some v_1 and $v_2 \in \mathcal{U}_*^{\geq QO}$. Since $p_1 + p_2 = q_1 + q_2$ we then have:

$$v_1 + v_2 = p_1 + p_2 - \theta_1^1 p_1 - \theta_2^1 p_2 - \theta_1^2 p_1 - \theta_2^2 p_2 = (1 - \theta_1^1 - \theta_1^2)(p_1 - p_2). \tag{14}$$

Now, since neither $p_2 - p_1 \in \mathcal{U}_*^{\geq QO}$ nor $p_1 - p_2 \in \mathcal{U}_*^{\geq QO}$ while $v_1 + v_2 \in \mathcal{U}_*^{\geq QO}$, the only way by which Equality (14) can hold is if $(1 - \theta_1^1 - \theta_1^2) = 0$ and, as a result, $v_1 + v_2 = 0$. Setting in that case $\theta_1 = \theta_1^1 = \theta_1^2$, we must therefore have $q_1 = \theta_1 p_1 + (1 - \theta_1) p_2;\ q_2 = (1 - \theta_1) p_1 + \theta_1 p_2$, so that $\mathbf{q} = \mathbf{m}.\mathbf{p}$, where $\mathbf{m} = \begin{bmatrix} \theta_1 & 1 - \theta_1 \\ 1 - \theta_1 & \theta_1 \end{bmatrix}$ is bistochastic.

[11] The case where $p_1 - p_2 \in \mathcal{U}_*^{\geq QO}$ is similar.

(Non-for-Publication) Online Appendix

Evaluating allocations of opportunities

A Axiomatic characterization of UEVEU rankings

We provide here a brief description of the axioms that happen to characterize the UEVEU family of orderings of allocations of opportunities. We start by giving additional notations. If $\mathbf{p}$ is an allocation of opportunities in $(\Delta^{k-1})^m$ and $\mathbf{q}$ is an allocation of opportunities in $(\Delta^{k-1})^n$, we denote by $(\mathbf{p}, \mathbf{q})$ the allocation of opportunities in $(\Delta^{k-1})^{m+n}$ where the m first groups have the opportunities associated with $\mathbf{p}$ and the n last groups have the opportunities associated with $\mathbf{q}$ (in the corresponding order). If ρ is in Δ^{k-1}, we abuse notation by also denoting by ρ the *one group allocation* in which people in the considered group face the opportunities $\rho \in \Delta^{k-1}$. We similarly sometimes abuse notation by using j to denote both the outcome $j \in \{1, ..., k\}$ itself and the degenerate probability distribution $\rho \in \Delta^{k-1}$ defined by $\rho_j = 1$ and $\rho_h = 0$ for all outcomes $h \neq j$ that gives j for sure.

The first axiom that enters in the characterization of the family of UEVEU family of criteria is the *anonymity* principle according to which the names of the groups don't matter for appraising the opportunities offered to their members. Hence, any between-group permutation of the probability distributions is a matter of indifference.

Axiom A1 *(Anonymity) $\pi.\mathbf{p} \sim \mathbf{p}$ must hold for every allocation of opportunities $\mathbf{p} \in \mathbb{A}$ and every $n(\mathbf{p}) \times n(\mathbf{p})$ permutation matrix π.*

While Anonymity seems plausible when evaluating allocations of opportunities among groups formed on the basis of race, gender, and other (morally arbitrary) qualitative characteristics of that sort, it may not seem so when groups are formed on the basis of a more quantitative attribute like, for example, the income category of the parents.

In such a setting, often considered in mobility measurement (see e.g. Atkinson (1981) and Dardanoni (1993)), it has been suggested that it could be better to give the good opportunities to kids from low-income families and the bad opportunities to kids from high-income family. In such a case, permuting the opportunities offered to children coming from different backgrounds is not be a matter of social indifference.

The second axiom is a continuity condition that concerns the ranking of opportunities faced by one group *vis-à-vis* others. This axiom requires that the strict ranking of any one-group allocation of opportunities *vis-à-vis* any other be robust to small changes in the probabilities of achieving any given outcome. Its formal statement is as follows.

Axiom A2 *(Continuity) For every allocation of opportunities* $\mathbf{p} \in \mathbb{A}$*, the sets* $B(\mathbf{p}) = \{\rho \in \Delta^{k-1} : \rho \succsim \mathbf{p}\}$*,* $W(p) = \{\rho \in \Delta^{k-1} : \mathbf{p} \succeq \rho\}$ *are both closed in* $\mathbb{R}^k_+$*.*

The third axiom is called *averaging* in Gravel, Marchant, and Sen (2012). It is the only axiom that restricts the ranking of allocations of opportunities among different numbers of groups. Specifically, the axiom evaluates what happens when two allocations of opportunities are merged together. To illustrate this, consider the allocations of educational opportunities in West Bengal and Odisha discussed in the introduction and their states' partition between low- and high-caste adults. Suppose, as suggested earlier, that educational opportunities are considered more equally distributed between low and high caste adults in West Bengal than in Odisha. Assume that the two states merge into a larger jurisdictional entity where there will now be four groups: Odisha low caste, Odisha high caste, West Bengal low caste and West Bengal high caste. The averaging axiom requires the ranking of the opportunities offered to this enlarged four-group jurisdiction to lie between that of the two initial two-groups states (West Bengal and Odisha). That is, opportunities should be better distributed in West Bengal than in the newly enlarged jurisdiction, and should be better distributed in this enlarged jurisdiction than they were in Odisha. The formal statement of this axiom is as follows.

Axiom A3 *(Averaging) For all allocations* $\mathbf{p}$ *and* $\mathbf{q}$ *in* $\mathbb{A}$*, we have* $\mathbf{p} \succsim \mathbf{q} \Leftrightarrow \mathbf{p} \succsim (\mathbf{p}, \mathbf{q}) \Leftrightarrow (\mathbf{p}, \mathbf{q}) \succsim \mathbf{q}$.

The next axiom requires the ranking of any two allocations of opportunities among the same number of groups to be robust to the addition, to the two allocations, of a common allocation of opportunities. Said differently, the ranking of two allocations of opportunities among the same number of groups should not depend upon any common allocation of opportunities among some of the groups.

Axiom A4 *(Same Number Group Independence) For all allocations* $\mathbf{p}$*,* $\mathbf{p}'$ *and* $\mathbf{p}''$ *in* $\mathbb{A}$ *such that* $n(\mathbf{p}) = n(\mathbf{p}')$*,* $(\mathbf{p}, \mathbf{p}'') \succsim (\mathbf{p}', \mathbf{p}'')$ *if and only if* $\mathbf{p} \succsim \mathbf{p}'$*.*

The last two axioms deal with the ranking of one-group allocations where by definition there is no concern for inequality of opportunities. The first of these axioms requires the ranking of one-group allocations, which is simply the ranking of probability distributions, to obey the well-known Von Neumann and Morgenstern (1947) independence axiom.

Axiom A5 *(VNM for One-Group) For every probability distributions* p*,* p' *and* p" $\in \Delta^{k-1}$ *and every number* $\lambda \in [0, 1]$*,* $p \succsim p'$ *if and only if* $\lambda p + (1 - \lambda)p'' \succsim \lambda p' + (1 - \lambda)p''$*.*

The second of these two axioms ensures the consistency of the ranking of one-group allocations, at least when the members of the group face no uncertainty at all, with the (possibly incomplete) ranking of outcomes provided by $\geq_{QO}$.

Axiom A6 *(Consistency with* $\geq_{QO}$ *for One-Group) For every two distinct outcomes* h *and* $j \in \{1, ..., k\}$ *such that* $j \geq_{QO} h$*, one should have* $j \succsim h$*.*

It can be checked that any ordering as per (2) satisfies Axioms A1 - A6. Using and adapting results in Gravel, Marchant, and Sen (2012) and Gravel, Marchant, and Sen (2011), the converse implication can be established. Hence, we have:

Proposition A1 *Let* $\succsim$ *be an ordering on* $\mathbb{A}$ *satisfying Axioms A1 - A6. Then, there exists a function* $\Phi : \mathbb{R} \to \mathbb{R}$ *and a list of* k *numbers* $u_1, ..., u_k$ *satisfying, for every two distinct outcomes* h *and* i*,* $i \geq_{QO} h \implies u_i \geq u_h$ *such that (2) holds for any two allocations* $\mathbf{p}$ *and* $\mathbf{q}$ *in* $\mathbb{A}$*. Furthermore, the function* Φ *is unique up to a positive affine transformation, and is continuous and increasing.*

B Additional proofs

B.1 Remark 1

Let $\mathbf{C}$ be the vector sub-space of $\mathbb{R}^k$, generated by the vector $(1,...,1)$. Observe that $\mathbf{C}$ is a convex cone, and is contained in $\mathcal{U}^{\geq_{QO}}$. Hence, by standard results, $\mathcal{U}_*^{\geq_{QO}} \subset \mathbf{C}_* = \left\{(v_1,...,v_k) \in \mathbb{R}^k : \sum_{j=1}^k v_k = 0\right\}$.

B.2 Lemma 1

Let $\theta_1,...,\theta_n \in [0,1]^n$, and suppose without loss of generality (thanks to anonymity) that $\theta_1 \leq \theta_2 \leq ... \leq \theta_n$. Then

$$\sum_{i=1}^n \theta_i p_i = \theta_1 \sum_{i=1}^n p_i + \sum_{l=2}^n \left((\theta_l - \theta_{l-1}) \sum_{i=l}^n p_i \right) + (1-\theta_l)\mathbf{0}.$$

The right-hand side being a convex combination of the set $\left\{\mathbf{0}, \left(\sum_{i=l}^n p_i\right)_{l=1,...,n}\right\}$, we get the result.

B.3 Remark 2

Suppose that $\mathbf{q} \succsim_Z^{QO} \mathbf{p}$ and, as a result, that $\mathbf{Z}(\mathbf{q}) + \mathcal{U}_*^{\geq_{QO}} \subset \mathbf{Z}(\mathbf{p}) + \mathcal{U}_*^{\geq_{QO}}$. Since in particular $\sum_{i=1}^n q_i \in \mathbf{Z}(\mathbf{q}) + \mathcal{U}_*^{\geq_{QO}}$, there is a collection of n numbers $\theta_1,...,\theta_n$ in the $[0,1]$ interval and a vector $v \in \mathcal{U}_*^{\geq_{QO}}$ such that $\sum_{i=1}^n q_i = \sum_{i=1}^n \theta_i p_i + v$, or, writing this equality for outcome j :$\sum_{i=1}^n q_{ij} = \sum_{i=1}^n \theta_i p_{ij} + v_j$. Summing over all outcomes, and exploiting the fact that $\sum_{j=1}^k v_j = 0$ and $\sum_{j=1}^k p_{ij} = \sum_{j=1}^k q_{ij} = 1$ for any i) one has:

$$\sum_{j=1}^n \sum_{i=1}^n q_{ij} = n = \sum_{i=1}^n \theta_i \sum_{j=1}^n p_{ij} + \sum_{j=1}^n v_j = \sum_{i=1}^n \theta_i$$

which implies that $\theta_i = 1$ for all i. Hence $\sum_{i=1}^n q_i = \sum_{i=1}^n p_i + v$, and $\sum_{i=1}^n q_i - \sum_{i=1}^n p_i = v \in \mathcal{U}_*^{\geq_{QO}}$, as required.

B.4 Remark 3

Observing that the inequality:

$$\sum_{i=1}^{n} \Phi \left(\sum_{h=1}^{k} q_{ih} u_h \right) \geq \sum_{i=1}^{n} \Phi \left(\sum_{h=1}^{k} p_{ih} u_h \right)$$

holds for all concave Φ and all lists of real numbers $u_1, ..., u_k$ is equivalent, thanks to the Hardy-Littlewood-Polya theorem, to the requirement that the matrix $\mathbf{q}$ price majorizes (using Kolm (1977) terminology) the matrix $\mathbf{p}$ for all price vectors $(u_1, ..., u_k)$. Koshevoy (1995) (Theorem 1) proves that the fact for a matrix $\mathbf{q} \in \mathbb{R}^{nd}$ to price majorize a matrix $\mathbf{p} \in \mathbb{R}^{nd}$ is equivalent to observing:

$$\begin{aligned}
\overline{Z}(\mathbf{q}) &= \left\{ \mathbf{z} \in \mathbb{R}^{k+1} : \; \mathbf{z} = \sum_{i=1}^{n} \theta_i \left(\frac{1}{n}, q_{i1}, ..., q_{ik} \right), \; \theta_i \in [0,1] \; \forall i = 1, \ldots, n \right\} \\
&\subseteq \left\{ \mathbf{z} \in \mathbb{R}^{k+1} : \; \mathbf{z} = \sum_{i=1}^{n} \theta_i \left(\frac{1}{n}, p_{i1}, ..., p_{ik} \right), \; \theta_i \in [0,1] \; \forall i = 1, \ldots, n \right\} \\
&= \overline{Z}(\mathbf{p})
\end{aligned}$$

Observe that the set $\overline{\mathbf{Z}}(\mathbf{a})$ (for any matrix $\mathbf{a} \in \mathbb{R}^{nd}$) defined in Koshevoy (1995) is somewhat similar to the set defined in Equation 5 above, with the exception that it takes the Minkowski sums over the population share extended vectors $(\frac{1}{n}, p_{i1}, ..., p_{ik})$ rather than over the vectors $(p_{i1}, ..., p_{ik})$ themselves. Hence we only need to prove that $\overline{\mathbf{Z}}(\mathbf{q}) \subseteq \overline{\mathbf{Z}}(\mathbf{p})$ is equivalent to $\mathbf{Z}(\mathbf{q}) \subseteq \mathbf{Z}(\mathbf{p}) \iff (\mathbf{Z}(\mathbf{q}) + \mathcal{U}_*^{\geq \emptyset}) \subseteq (\mathbf{Z}(\mathbf{p}) + \mathcal{U}_*^{\geq \emptyset})$ to complete the argument. The fact that $\overline{\mathbf{Z}}(\mathbf{q}) \subseteq \overline{\mathbf{Z}}(\mathbf{p})$ implies $\mathbf{Z}(\mathbf{q}) \subseteq \mathbf{Z}(\mathbf{p})$ is obvious. To establish the other direction, assume that $\mathbf{Z}(\mathbf{q}) \subseteq \mathbf{Z}(\mathbf{p})$. This means that for any list of numbers $\theta_1, ..., \theta_n$ in the $[0,1]$ interval, one can find a list of numbers $\theta'_1, ..., \theta'_n$ in the $[0,1]$ interval such that $\sum_{i=1}^{n} \theta_i q_i = \sum_{i=1}^{n} \theta'{}_i p_i$. Observe that this equality implies that for any $j = 1, ..., k$ one has: $\sum_{i=1}^{n} \theta_i q_{ij} = \sum_{i=1}^{n} \theta'{}_i p_{ij}$. Summing these equalities over all j yields (exploiting the fact that the probability distributions lie in Δ^{k-1}):

$$\sum_{i=1}^{n} \theta_i \sum_{j=1}^{k} q_{ij} = \sum_{i=1}^{n} \theta_i = \sum_{i=1}^{n} \theta'{}_i \sum_{j=1}^{k} q_{ij} = \sum_{i=1}^{n} \theta'{}_i$$

But this implies that for any for any list of numbers $\theta_1, ..., \theta_n$ in the $[0,1]$ interval, one can find a list of numbers $\theta'_1, ..., \theta'_n$ in that same interval such that;

$$\sum_{i=1}^{n} \theta_i \left(\frac{1}{n}, q_{i1}, ..., q_{ik}\right) = \sum_{i=1}^{n} \theta'_i \left(\frac{1}{n}, p_{i1}, ..., p_{ik}\right)$$

That is, this implies that $\overline{\mathbf{Z}}(\mathbf{q}) \subseteq \overline{\mathbf{Z}}(\mathbf{p})$ holds, as required. The fact that $\overline{p} = \overline{q}$ is an immediate consequence of Remark 2 and the fact that $\mathcal{U}_*^{\geq^\emptyset} = \{\mathbf{0}^k\}$.

B.5 Lemma 2

The fact that

$$\mathcal{U}_*^{\geq_{QO}} \subseteq \left\{(v_1, ..., v_k) \in \mathbb{R}^k : \sum_{j=1}^{k} v_j u_j \geq 0 \ \forall (u_1, ..., u_k) \in \mathcal{U}^{\geq_{QO}} \cap \{0,1\}^k\right\}$$

directly follows from the fact that $\mathcal{U}^{\geq_{QO}} \cap \{0,1\}^k \subset \mathcal{U}^{\geq_{QO}}$.

To prove the reverse inclusion, consider any $(v_1, ..., v_k)$ satisfying $\sum_{j=1}^{k} v_j u_j \geq 0$ for all $(u_1, ..., u_k) \in \mathcal{U}^{\geq_{QO}} \cap \{0,1\}^k$. We must show that it satisfies also $\sum_{j=1}^{k} v_j u_j \geq 0$ for any $(u_1, ..., u_k) \in \mathcal{U}^{\geq_{QO}}$. Consider therefore any such $(u_1, ..., u_k) \in \mathcal{U}^{\geq_{QO}}$. By continuity of the map $(u_1, ..., u_k) \mapsto \sum_{j=1}^{k} v_j u_j$, we may assume without loss of generality that $u_h \neq u_i$ for any two distinct h and i in $\{1, ..., k\}$. Let $j : \{1, ..., k\} \longrightarrow \{1, ..., k\}$ be a one-to-one function such that such that $u_{j(1)} < u_{j(2)} < ... < u_{j(k)}$. We have:

$$\sum_{j=1}^{k} v_j u_j = \sum_{h=1}^{k} v_{j(h)} u_{j(h)} = \sum_{h=2}^{k} v_{j(h)} (u_{j(h)} - u_{j(1)}) \tag{B.1}$$

since $v_1 + ... + v_k = 0$.Using Abel decomposition formula, one can alternatively write this equality as:

$$\sum_{j=1}^{k} v_j u_j = \sum_{h=2}^{k} (u_{j(h)} - u_{j(h-1)}) \sum_{g=h}^{k} v_{j(g)}$$

Now, for any $h = 2, ..., k$, let $w^h \in \{0,1\}^k$ be defined by: $w^h_{j(g)} = \begin{cases} 0 & \text{if} \quad g < h, \\ 1 & \text{if} \quad g \geq h. \end{cases}$
We observe that, for any $h \in \{2, ..., k\}$, $(w^h_{j(1)}, ..., w^h_{j(k)}) \in \mathcal{U}^{\geq_{QO}}$. Indeed, if $l >^{QO} g$ for

two distinct outcomes g and l in $\{1, ..., k\}$, then $u_l > u_g$ by definition of $(u_1, ..., u_k) \in \mathcal{U}^{\geq_{QO}}$. Given h, three cases are possible:

(i) $l < h$. In this case, one has $w^h_{j(g)} = 0 = w^h_{j(l)}$ from the definition of w^h;

(ii) $g < h \leq l$. In this case, $w^h_{j(g)} = 0 < 1 = w^h_{j(l)}$ holds from the definition of w^h and the required weak inequality $w^h_{j(g)} \leq w^h_{j(l)}$ is also satisfied;

(iii) $h \leq g < l$. In this case $w^h_{j(g)} = 1 = w^h_{j(l)}$ holds from the definition of w^h.

Hence, in all the three cases, the required weak inequality$w^h_{j(g)} \leq w^h_{j(l)}$ is satisfied. Since $(w^h_{j(1)}, ..., w^h_{j(k)}) \in \mathcal{U}^{\geq_{QO}} \cap \{0,1\}^k$ for any $h = 2, ..., k$, we have $\sum_{g=1}^k v_{j(g)} w^h_{j(g)} = \sum_{g=h}^k v_{j(g)} \geq 0$ for any such h. But this implies that $\sum_{h=2}^k v_{j(h)}(u_{j(h)} - u_{j(1)}) \geq 0$ for any such h which, thanks to Equality (B.1), establishes the result.

B.6 Lemma 3

For uniform averaging, we simply observe that the function $\Psi : \Delta^{k-1} \to \mathbb{R}$ defined, for every $(s_1, ..., s_k) \in \Delta^{k-1}$ by:

$$\Psi(s_1, ..., s_k) = \Phi\left(\sum_{h=1}^k s_h u_h\right)$$

is concave if Φ is concave irrespective of what the real numbers $(u_1, ..., u_k)$ are. Hence, by virtue of Theorem 3 in Kolm (1977),

$$\sum_{i=1}^n \Phi\left(\sum_{h=1}^k q_{ih} u_h\right) \geq \sum_{i=1}^n \Phi\left(\sum_{h=1}^k p_{ih} u_h\right)$$

if there exists a bistochastic matrix $n \times n$ bistochastic matrix $\mathbf{b}$ such that $\mathbf{q} = \mathbf{b}.\mathbf{p}$. Assume now that $(u_1, ..., u_k) \in \mathcal{U}^{\geq_{QO}}$ and that $\mathbf{q}$ results from from $\mathbf{p}$ through an equalizing transfer as per Definition 6. We must show that:

$$\sum_{i=1}^n \Phi\left(\sum_{h=1}^k q_{ih} u_h\right) \geq \sum_{i=1}^n \Phi\left(\sum_{h=1}^k p_{ih} u_h\right).$$

Since all rows others than i_1 and i_2 in the matrix $\mathbf{p}$ and others than i_1' and i_2' in the matrix $\mathbf{q}$ are unaffected by the change, we have:

$$\sum_{i=1}^{n} \Phi\left(\sum_{h=1}^{k} q_{ih} u_h\right) \geq \sum_{i=1}^{n} \Phi\left(\sum_{h=1}^{k} p_{ih} u_h\right)$$

$$\Longleftrightarrow$$

$$\Phi\left(\sum_{h=1}^{k} q_{i_1' h} u_h\right) + \Phi\left(\sum_{h=1}^{k} q_{i_2' h} u_h\right) \geq \Phi\left(\sum_{h=1}^{k} p_{i_1 h} u_h\right) + \Phi\left(\sum_{h=1}^{k} p_{i_2 h} u_h\right) \tag{B.2}$$

We now observe that

$$\left(\sum_{h=1}^{k} q_{i_1' h} u_h, \sum_{h=1}^{k} q_{i_2' h} u_h\right) \text{ Lorenz-dominates } \left(\sum_{h=1}^{k} p_{i_1 h} u_h, \sum_{h=1}^{k} p_{i_2 h} u_h\right).$$

Indeed, one has:

$$\sum_{h=1}^{k} p_{i_1 h} u_h \leq \sum_{h=1}^{k} p_{i_2 h} u_h - \sum_{h=1}^{k} v_h u_h = \sum_{h=1}^{k} q_{i_2' h} u_h \leq \sum_{h=1}^{k} p_{i_2 h} u_h$$

and:

$$\sum_{h=1}^{k} p_{i_1 h} u_h \leq \sum_{h=1}^{k} p_{i_1 h} u_h + \sum_{h=1}^{k} v_h u_h = \sum_{h=1}^{k} q_{i_1' h} u_h \leq \sum_{h=1}^{k} p_{i_2 h} u_h$$

Inequality (B.2) then follows from the Hardy-Littlewood-Polya Theorem.

B.7 Lemma 4

As noted in the proof of Theorem 1 (before Expression (8)), $\mathbf{q} \succsim_Z^{QO} \mathbf{p}$ if and only if, for all $\alpha_1, ..., \alpha_n \in \{0,1\}^n$, there exist $v \in \mathcal{U}_*^{\geq QO}$ and $\theta_1, ..., \theta_n \in [0,1]^n$ such that $\sum_{i=1}^n \alpha_i q_i = \sum_{i=1}^n \theta_i p_i + v$. This is in turn equivalent to having that, for all $h = 1, \ldots, n$ and all $m = 1, \ldots, m(h)$, there exist $v \in \mathcal{U}_*^{\geq QO}$ and $\theta_1, ..., \theta_n \in [0,1]^n$ such that $\sum_{i=1}^n \theta_i = h$ and $\sum_{i \in J_h^m} q_i = \sum_{i=1}^n \theta_i p_i + v$. Proving the lemma therefore amounts to showing that, for any $h = 1, ..., n$, we have

$$\left\{\sum_{i=1}^{n} \theta_i p_i : \theta_1, ..., \theta_n \in [0,1]^n, \ \sum_{i=1}^{n} \theta_i = h\right\} = Co\left\{\sum_{i \in J_h^1} p_i, \ldots, \sum_{i \in J_h^{m(h)}} p_i\right\} \tag{B.3}$$

The set on the left-hand side of the equality is the intersection of a convex polytope, $\mathbf{Z}(\mathbf{p})$, and an affine subspace of $\mathbb{R}^k$. Hence it is also a convex polytope. Since it contains the points $\sum_{i\in J_h^1} p_i, \ldots, \sum_{i\in J_h^{m(h)}} p_i$, we have:

$$Co\left\{\sum_{i\in J_h^1} p_i, \ldots, \sum_{i\in J_h^{m(h)}} p_i\right\} \subset \left\{\sum_{i=1}^n \theta_i p_i : \theta_1, ..., \theta_n \in [0,1]^n,\ \sum_{i=1}^n \theta_i = h\right\}.$$

Let $x^1, ..., x^P$ be the extreme points of $\{\sum_{i=1}^n \theta_i p_i : \theta_1, ..., \theta_n \in [0,1]^n,\ \sum_{i=1}^n \theta_i = h\}$. Let us now show that

$$x^p \in \left\{\sum_{i\in J_h^1} p_i, \ldots, \sum_{i\in J_h^{m(h)}} p_i\right\},\ \forall p = 1, ..., P.$$

Suppose by contradiction that this is not the case. Then $x^p = \sum_{i=1}^n \theta_i p_i$, where we can assume without loss of generality that $\theta_1 \in]0,1[$. Since $\sum_{i=1}^n \theta_i = h$ there must exist another real number θ_2 such that $\theta_2 \in]0,1[$. We then have $x^p = \frac{1}{2}(x_+ + x_-)$, where $x_+ := (\theta_1+\varepsilon)p_1 + (\theta_2-\epsilon)p_2 + ... + \theta_n p_n$ and $x_- := (\theta_1-\epsilon)p_1 + (\theta_2+\epsilon)p_2 + ... + \theta_n p_n$ both belong to $\{\sum_{i=1}^n \theta_i p_i : \theta_1, ..., \theta_n \in [0,1]^n,\ \sum_{i=1}^n \theta_i = h\}$, provided that ε is small enough. Hence x^p is not an extreme point of $\{\sum_{i=1}^n \theta_i p_i : \theta_1, ..., \theta_n \in [0,1]^n,\ \sum_{i=1}^n \theta_i = h\}$, a contradiction. Hence (B.3) holds, and this concludes the proof.

B.8 Proposition 2

If $\mathbf{q} \succsim_Z^{QO} \mathbf{p}$ then there exists some $\tilde{p} \in Co\left\{\sum_{i\in J_h^1} p_i, \ldots, \sum_{i\in J_h^{m(h)}} p_i\right\}$ such that

$$\frac{1}{h}\sum_{i=1}^h q_i - \frac{1}{h}\tilde{p} \in \mathcal{U}_*^{\geq QO},$$

by Lemma 4. Since $\frac{1}{h}\tilde{p} - \frac{1}{h}\sum_{i=1}^h p_i \in \mathcal{U}_*^{\geq QO}$ we get Expression (6) of Proposition 2. Suppose now that Expression (6) holds. Since $\frac{1}{h}\sum_{i\in J_h^m} q_i - \frac{1}{h}\sum_{i=1}^h q_i \in \mathcal{U}_*^{\geq QO}$, we have:

$$\frac{1}{h}\sum_{i\in J_h^m} q_i - \frac{1}{h}\sum_{i=1}^h p_i = \frac{1}{h}\sum_{i\in J_h^m} q_i - \frac{1}{h}\sum_{i=1}^h q_i + \frac{1}{h}\sum_{i=1}^h q_i - \frac{1}{h}\sum_{i=1}^h p_i \in \mathcal{U}_*^{\geq QO}$$

and this concludes the proof.

B.9 Remark 4.

Let the transformed allocation $\mathbf{p}'$ be defined by $p_1' = p_1 + w_1$ and $p_2' = p_2 + w_2$ for some $w_1, w_2 \in \mathcal{U}_*^{\geq C}$. We claim that if $\mathbf{q} \succsim_Z^C \mathbf{p}'$ then $w_1 + w_2 = 0$. Suppose indeed that: $q_1 - (\theta_1 p_1' + (1-\theta_1)p_2') \in \mathcal{U}_*^{\geq C}$, $q_2 - (\theta_2 p_1' + (1-\theta_2)p_2') \in \mathcal{U}_*^{\geq C}$, and $q_2 + q_1 - (p_1' + p_2') \in \mathcal{U}_*^{\geq C}$. Then it follows that $\theta_1 = 1$, as we have seen in the argument that we just made about the impossibility of performing a uniform averaging. This implies that $q_1 - p_1 - w_1 \in \mathcal{U}_*^{\geq C}$, that is $\frac{1}{36}(0,-2,2,0) - w_1 \in \mathcal{U}_*^{\geq C}$. Secondly

$$q_2 - (\theta_2 p_1 + (1-\theta_2)p_2) = \frac{1}{36}\left(-3\theta_2, 2-\theta_2, -3+6\theta_2, 1-2\theta_2\right).$$

This vector belongs to $\mathcal{U}_*^{\geq C}$ if and only if $\theta_2 = 1/2$ and it is then equal to $\frac{1}{72}\left(-3,3,0,0\right)$. To sum up we have:

$$\frac{1}{36}(0,-2,2,0) - w_1 \in \mathcal{U}_*^{\geq C},\ \ \frac{1}{72}\left(-3,3,0,0\right) - \frac{1}{2}(w_1 + w_2) \in \mathcal{U}_*^{\geq C}$$

and:

$$\frac{1}{36}\left(0,0,-1,1\right) - (w_1 + w_2) \in \mathcal{U}_*^{\geq C}.$$

Now $w_1 + w_2 = (a,b,c,d)$ is by assumption an element of $\mathcal{U}_*^{\geq C}$. The condition $\frac{1}{36}\left(-3,3,0,0\right) - (w_1 + w_2) \in \mathcal{U}_*^{\geq C}$ implies that $c = d = 0$. On the other hand the condition $\frac{1}{36}\left(0,0,-1,1\right) - (w_1 + w_2) \in \mathcal{U}_*^{\geq C}$ implies that $a = b = 0$. Thus $w_1 + w_2 = 0$ and, actually, $w_1 = w_2 = 0$.